QCD factorization for quasi-parton distributions at twist-three level

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This talk is based on

- [2103.12105] (*JHEP 05 (2021) 086*) V.Braun, Y. Ji, AV
  "QCD factorization for twist-three axial-vector parton quasidistributions"
- [2108.03065] V.Braun, Y. Ji, AV
  "QCD factorization for chiral-odd parton quasi- and pseudo-distributions"

I am going to present you the factorization theorem for twist-3 quasidistributions and discuss how one can extract twist-3 distributions from them.

Outline

- What are twist-three distributions
- Quasi Ioffe-time distributions (qITDs) and factorization theorem for them
- From qITDs to pPDFs and qPDFs
- How to access twist-3 distribution

Disclaimer:
It is a purely theoretical talk ⇒ many formulas ⇒ easy to get lost
Ask questions!

Everyone knows twist-2 distributions: $\{q,\ \delta q,\ \Delta q\}$ or $\{f_1,\ g_1,\ h_1\}$
What about twist-3 distributions?

- There is a terminological ambiguity.

[Jaffe,Ji,Nucl.Phys.B 92]

$$\int \frac{d\lambda}{2\pi} \; e^{i\lambda x} \langle PS|\bar{\psi}(0)\gamma_\mu\gamma_5\psi(\lambda n)|PS\rangle$$
$$\equiv 2[g_1(x)p_\mu(S\cdot n) + g_T(x)S_{\perp\mu} + M^2 g_3(x) n\cdot S n_\mu], \qquad (3)$$

where we have written $S_\mu = S\cdot n p_\mu + S\cdot p n_\mu + S_{\perp\mu}$. These distribution functions, $g_1(x)$, $g_T(x) \equiv g_1(x) + g_2(x)$, and $g_3(x)$, contribute to a hard process at order $g_1(x), g_T(x)/Q, g_3(x)/Q^2$ and hence are twist-two, three and four, respectively. Typically a distribution function of twist-t appears in physical observables with coefficient powers of $(1/Q)^{t-2}$, $(1/Q)^{t-1}$, etc. Thus there appears to be some ambiguity in the definition of higher-twist $(t > 2)$ structure functions. For example, either $g_2(x)$ or $g_T(x) = g_1(x) + g_2(x)$ can be a useful measure of transverse spin in deep inelastic scattering depending on the circumstances. A light-cone operator product expansion provides a *formal* and powerful way of isolating "irreducibly" higher-twist parts of higher-twist structure functions. This method will be applied to cases of interest in subsect.

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What about twist-3 distributions?

[Jaffe,Ji,Nucl.Phys.B 92]

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\gamma_\mu \psi(\lambda n)|PS\rangle \equiv 2\left[f_1(x)p_\mu + M^2 f_4(x)n_\mu\right]$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\gamma_\mu\gamma_5 \psi(\lambda n)|PS\rangle$$
$$\equiv 2[g_1(x)p_\mu(S\cdot n) + g_T(x)S_{\perp\mu} + M^2 g_3(x) n\cdot S n_\mu]$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\psi(\lambda n)|PS\rangle \equiv 2Me(x)$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\sigma_{\mu\nu} i\gamma_5 \psi(\lambda n)|PS\rangle \equiv 2\Big[h_1(x)(S_{\perp\mu}p_\nu - S_{\nu\perp}p_\mu)/M$$
$$+h_L(x)M(p_\mu n_\nu - p_\nu n_\mu)(S\cdot n)$$
$$+h_3(x)M(S_{\perp\mu}n_\nu - S_{\perp\nu}n_\mu)\Big], \quad (5)$$

Everyone knows twist-2 distributions: $\{q,\ \delta q,\ \Delta q\}$ or $\{f_1,\ g_1,\ h_1\}$
What about twist-3 distributions?

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$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar\psi(0)\gamma_\mu\psi(\lambda n)|PS\rangle \equiv 2\left[f_1(x)p_\mu + M^2 f_4(x) n_\mu\right] \quad \textbf{twist-2}$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar\psi(0)\gamma_\mu\gamma_5\psi(\lambda n)|PS\rangle \equiv 2\left[g_1(x)p_\mu(S\cdot n) + g_T(x)S_{\perp\mu} + M^2 g_3(x) n\cdot S n_\mu\right]$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar\psi(0)\psi(\lambda n)|PS\rangle \equiv 2Me(x)$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar\psi(0)\sigma_{\mu\nu}i\gamma_5\psi(\lambda n)|PS\rangle \equiv 2\Big[h_1(x)(S_{\perp\mu}p_\nu - S_{\nu\perp}p_\mu)/M + h_L(x)M(p_\mu n_\nu - p_\nu n_\mu)(S\cdot n) + h_3(x)M(S_{\perp\mu}n_\nu - S_{\perp\nu}n_\mu)\Big], \quad (5)$$

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What about twist-3 distributions?

[Jaffe,Ji,Nucl.Phys.B 92]

twist-2

twist-3

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\gamma_\mu \psi(\lambda n)|PS\rangle \equiv 2\left[f_1(x)p_\mu + M^2 f_4(x) n_\mu\right]$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\gamma_\mu\gamma_5 \psi(\lambda n)|PS\rangle$$
$$\equiv 2\left[g_1(x)p_\mu(S\cdot n) + g_T(x) S_{\perp\mu} + M^2 g_3(x) n\cdot S n_\mu\right]$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\psi(\lambda n)|PS\rangle \equiv 2M e(x)$$

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\sigma_{\mu\nu} i\gamma_5 \psi(\lambda n)|PS\rangle \equiv 2\Big[h_1(x)(S_{\perp\mu}p_\nu - S_{\nu\perp}p_\mu)/M$$
$$+ h_L(x) M(p_\mu n_\nu - p_\nu n_\mu)(S\cdot n)$$
$$+ h_3(x) M(S_{\perp\mu}n_\nu - S_{\perp\nu}n_\mu)\Big], \quad (5)$$

I am going to discuss "quasi"-analogs of twist-3 $g_T(x)$, $e(x)$ and $h_L(x)$ in terms of **"formal"** twist-3 distributions

Formal definition of the twist ⇒ *geometrical twist= dimension - spin*
Defined for the local operators

$$\bar{q}(z)\gamma^{\mu}[z,0]q(0) = \sum_{n=0}^{\infty} \frac{z_{\mu_1}...z_{\mu_n}}{n!} O^{\mu\mu_1\cdots\mu_n}$$
$$O^{\mu_1\cdots\mu_n} = \bar{q}\gamma^{\mu_1} D^{\mu_2}...D^{\mu_n} q$$

A Lorentz tensor
Sort it over irr.rep.

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$$O^{\mu_1\cdots\mu_n} = \bar{q}\gamma^{\mu_1}D^{\mu_2}...D^{\mu_n}q$$

A Lorentz tensor
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$$O^{\mu\nu} = \underbrace{\left[\frac{O^{\mu\nu}+O^{\nu\mu}}{2} - \frac{g^{\mu\nu}}{4}O^{\rho}{}_{\rho}\right]}_{\text{spin}=2} + \underbrace{\left[\frac{O^{\mu\nu}-O^{\nu\mu}}{2}\right]}_{\text{spin}=1} + \underbrace{\frac{g^{\mu\nu}}{4}O^{\rho}{}_{\rho}}_{\text{spin}=0}$$

2-index example

Formal definition of the twist ⇒ *geometrical twist= dimension - spin*
Defined for the local operators

$$\bar{q}(z)\gamma^{\mu}[z,0]q(0)=\sum_{n=0}^{\infty}\frac{z_{\mu_1}...z_{\mu_n}}{n!}O^{\mu\mu_1\cdots\mu_n}$$
$$O^{\mu_1\cdots\mu_n}=\bar{q}\gamma^{\mu_1}D^{\mu_2}...D^{\mu_n}q$$

A Lorentz tensor
Sort it over irr.rep.

$$O^{\mu_1\cdots\mu_n}=\underbrace{\Big[\frac{O^{\mu_1\mu_2..\mu_n}+O^{\mu_2\mu_1..\mu_n}+...}{n!}-\frac{g^{\mu_1\mu_2}O^{\rho}{}_{\rho}{}^{\cdots\mu_n}}{4(n-2)!}+...\Big]}_{\substack{\text{totally simmetric traceless}\\ \text{spin=n}\\ \text{twist=2}}}$$

$$+\underbrace{\Big[\frac{O^{\mu_1\mu_2..\mu_n}-O^{\mu_2\mu_1..\mu_n}+...}{n!}+...\Big]}_{\substack{\text{1-pair-anti-symmetric + simmetric traceless}\\ \text{spin=n-1}\\ \text{twist=3}}}+$$

Twist-decomposition
is purely algebraic operation

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Non-local OPE

$z^2 \ll \Lambda^{-2}$

non-local operators: $\bar{q}(z)[z,0]\gamma^\mu q(0)$ — indefinite twist

⟵ Taylor series ⟶

local operators: $\sum_{n=0}^{\infty} \frac{z_{\mu_1}\ldots z_{\mu_n}}{n!} \bar{q}\overleftarrow{D}^{\mu_1}\ldots\overleftarrow{D}^{\mu_n}\gamma^\mu q(0)$ — indefinite twist

↓ Twist decomposition ↓ Twist decomposition

$\sum_{k=2}^{\infty} \mathbf{c}_k \otimes \underbrace{[O(z)]^{\text{tw}-k}}_{\text{definite twist}}$

⟵ Generating func. ⟶

$\sum_{n=0}^{\infty} \frac{z_{\mu_1}\ldots z_{\mu_n}}{n!} \sum_k c_{n,k}\, \underbrace{O_k^{\mu_1\ldots\mu_n}(0)}_{\text{definite twist} = \text{dim.-spin}}$

- $\mathbf{c}$ is an integro-differential operator
- There are methods to make twist-decomposition directly in the non-local form
- Famous example: $$[\bar{q}(z)[z,0]\gamma^\mu q(0)]^{\text{tw}-2} = \int_0^1 d\alpha \frac{\partial}{\partial z_\mu} \bar{q}(\alpha z)[\alpha z, 0]\not{z} q(0) + O(z^2)$$

[Balitsky,Braun,89]

Lorentz symmetry is preserved by renormalization
The **geometrical twist** (as a quantum number) is preserved by renormalization
Thus, operators of different geometrical twist do not mix with each other, and result into **independent physical observables**

Collinear (naive) twist

- No formal definition
- Usually, counted by dimension
- Examples: $g_T(x)$, $h_L(x)$, $e(x)$
- Do not have definite evolution properties
- Some of them have "parton interpretation"

Geometrical twist

- Formal definition
- Closed evolution equation
- Do not have a simple "parton interpretation"
- Almost unknown phenomenologically

Factorization theorem is formulate in terms of geometrical twist distribution

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There are <u>four</u> quark-PDFs of twist-three

invariant definition; $z^2 = 0$

$$\langle p| ig\, \bar q(az)\gamma^\rho \not z \gamma^\mu \gamma^5 F_{\rho z}(bz) q(cz)|p\rangle = 4 s_T^\mu \zeta^2 M \widehat{S}^+(a\zeta, b\zeta, c\zeta), \quad (1)$$

$$\langle p| ig\, \bar q(az)\gamma^\mu \not z \gamma^\rho \gamma^5 F_{\rho z}(bz) q(cz)|p\rangle = 4 s_T^\mu \zeta^2 M \widehat{S}^-(a\zeta, b\zeta, c\zeta), \quad (2)$$

$$\langle p| g\, \bar q(az)\sigma^{\mu z} \gamma^5 F_{\mu z}(bz) q(cz)|p\rangle = 4 \lambda_z \zeta^2 M \widehat{H}(a\zeta, b\zeta, c\zeta), \quad (3)$$

$$\langle p| g\, \bar q(az)\sigma^{\mu z} F_{\mu z}(bz) q(cz)|p\rangle = 4 \zeta^2 M \widehat{E}(a\zeta, b\zeta, c\zeta). \quad (4)$$

$$\zeta = (pz) = p_z = \text{Ioffe time}$$

There is no standard notation!

$$S^\pm = -T \mp \Delta T$$

$$H = \delta T_g$$

$$E = \delta T_\epsilon$$

....

N \ q	U	L	T
U	q		δT_ϵ
L		Δq	δT_g
T	T	ΔT	δq

There are <u>four</u> quark-PDFs of twist-three

light-cone definition, $z^\mu = n^\mu z^-$

$$\langle p| ig\,\bar q(az)\gamma^\rho\gamma^+\gamma^\mu\gamma^5 F_{\rho+}(bz)q(cz)|p\rangle = 4 s_T^\mu p_+^2 M \widehat{S}^+(a\zeta,b\zeta,c\zeta), \quad (1)$$

$$\langle p| ig\,\bar q(az)\gamma^\mu\gamma^+\gamma^\rho\gamma^5 F_{\rho+}(bz)q(cz)|p\rangle = 4 s_T^\mu p_+^2 M \widehat{S}^-(a\zeta,b\zeta,c\zeta), \quad (2)$$

$$\langle p| g\,\bar q(az)\sigma^{\mu z}\gamma^5 F_{\mu+}(bz)q(cz)|p\rangle = 4 s^+ p_+ M^2 \widehat{H}(a\zeta,b\zeta,c\zeta), \quad (3)$$

$$\langle p| g\,\bar q(az)\sigma^{\mu+} F_{\mu+}(bz)q(cz)|p\rangle = 4 p_+^2 M \widehat{E}(a\zeta,b\zeta,c\zeta). \quad (4)$$

$$\boxed{\zeta = (pz) = p_z = \text{Ioffe time}}$$

- There is no standard notation!
- $S^\pm$ are chiral even, H, E are chiral-odd
- Alike twist-2 but with additional $F_{\mu+}$ (due to anti-symmetrized $[D_\mu, D_\nu]$)
- Depend on 3 position $\Rightarrow$ 3-variable PDFs (vs. 2 at twist-2)
- 3 positions - 1 common position = 2 d.o.f. (for PDF!)

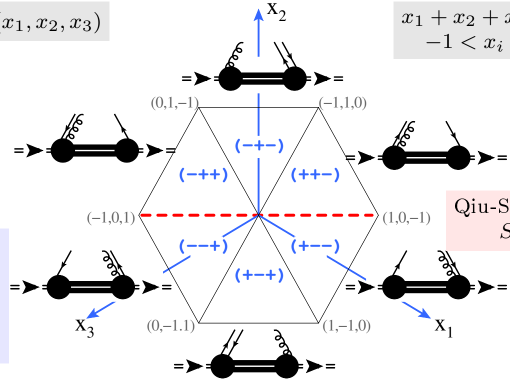
Notation comment: Widehat = position space, e.g. $\widehat{g_T}(\zeta) = \int_{-1}^{1} dx e^{ix\zeta} g_T(x)$.

$$\widehat{S}^{\pm}(\zeta_1,\zeta_2,\zeta_3) = \int [dx] e^{-i(\zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3)} S^{\pm}(x_1,x_2,x_3)$$

$$\int [dx] = \int_{-1}^{1} dx_1\, dx_2\, dx_3\, \delta(x_1+x_2+x_3)$$

$S^{\pm}(x_1,x_2,x_3)$

$x_1 + x_2 + x_3 = 0$
$-1 < x_i < 1$

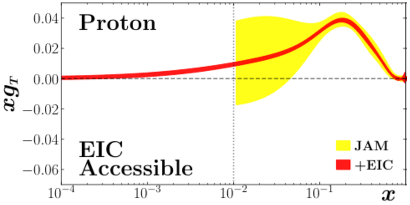


Qiu-Sterman function
$S^{\pm}(-x,0,x)$

Each region
$x_i \lessgtr 0$
has its own
partonic
interpretation

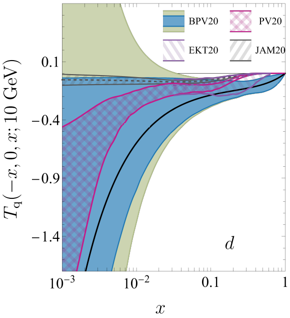
Not too much is known about twist-three distributions

From DIS (g_2 structure function)



[JAM20]

From TMDs



[Bury,Prokudin,AV,21]

Lattice input will be essential!

Quasi-Ioffe-time distributions (qITDs)
measurable on the lattice; $z^2 < 0$

$$\langle p,s|\bar{q}(z)\gamma^\mu q(0)|p,s\rangle = 2p^\mu \mathcal{F}_1(\zeta,z^2) + 2\frac{z^\mu\zeta - p^\mu z^2}{\zeta^2}M^2\mathcal{F}_3(\zeta,z^2)$$

$$\langle p,s|\bar{q}(z)\gamma^\mu\gamma^5 q(0)|p,s\rangle = 2p^\mu s_L\mathcal{G}_1(\zeta,z^2) + 2s_T^\mu M\mathcal{G}_T(\zeta,z^2) + \frac{z^\mu\zeta - p^\mu z^2}{\zeta^2}s_L M^2\mathcal{G}_3(\zeta,z^2)$$

$$\langle p,s|\bar{q}(z)i\sigma^{\mu z}\gamma^5 q(0)|p,s\rangle = 2s_T^\mu\zeta\mathcal{H}_1(\zeta,z^2) - (sz)\left(z^\mu - p^\mu\frac{z^2}{\zeta}\right)M\mathcal{H}_L(\zeta,z^2)$$

$$\langle p,s|\bar{q}(z)q(0)|p,s\rangle = 2M\mathcal{E}(\zeta,z^2)$$

s_T^μ **transverse to** (p,z)**-plane**

At leading power $z^2 \to 0$

- (green) = twist-2
- (blue) = twist-2 + twist-3 ← that is what we are interested in
- (red) = twist-2 + twist-3 + twist-4

Notation comment: $f_1 \to \mathcal{F}_1$, $g_T \to \mathcal{G}_T$, etc.

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qITDs (and position space) are much simpler from the theory side
(there will be examples in the talk).
pPDFs and qPDF are Fourier transformation of qITDs

$$\text{pPDF}(x, z^2) = \int \frac{d\zeta}{2\pi} e^{-ix\zeta} \text{qITD}(\zeta, z^2)$$

$$\text{qPDF}(x, p_v^2) = p_v \int \frac{dz}{2\pi} e^{-ixzp_v} \text{qITD}(zp_v, z^2)$$

Plan:

- Derive all formulas for qITDs (factorization at $z^2 \to 0$)
- Fourier transform to pPDF (factorization at $z^2 \to 0$)
- Fourier transform to qPDF (factorization at $p_v^2 \to \infty$)

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LO factorization, $z^2 \to 0$

Tree order is trivial

- Set $z^2 = 0$ and compare parametrizations (e.g. with Jaffe,Ji)

$$\begin{aligned} \mathcal{G}_T(\zeta, z^2) &= \widehat{g_T}(\zeta) + O(z^2) \\ \mathcal{H}_L(\zeta, z^2) &= \widehat{h_L}(\zeta) + O(z^2) \\ \mathcal{E}(\zeta, z^2) &= \widehat{e}(\zeta) + O(z^2) \end{aligned}$$

- But it is not enough, because:
  - At NLO the expression is not expressed via g_T, h_L or e
  - g_T, h_L and e are not "true" distributions (no evolution)
  - g_T, h_L have twist-2 part
- We must make twist-decomposition

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LO factorization, $z^2 \to 0$

Tree order is not so trivial

- Symmetrizing, anti-symmetrizing,...

$$\begin{aligned}
\mathcal{G}_T(\zeta, z^2) &= \widehat{g}_T^{\,\mathrm{tw}\,2}(\zeta) + \widehat{g}_T^{\,\mathrm{tw}\,3}(\zeta) + O(z^2) \\
\mathcal{H}_L(\zeta, z^2) &= \widehat{h}_L^{\,\mathrm{tw}\,2}(\zeta) + \widehat{h}_L^{\,\mathrm{tw}\,3}(\zeta) + O(z^2) \\
\mathcal{E}(\zeta, z^2) &= \Sigma_q + \widehat{e}_{\mathrm{nl}}(\zeta) + O(z^2)
\end{aligned}$$

- Twist-2 parts (here $x > 0$)

$$\widehat{g}_T^{\,\mathrm{tw}\,2}(\zeta) = \int_0^1 d\alpha\, \widehat{\Delta q}(\alpha\zeta), \qquad \Leftrightarrow \qquad g_T^{\mathrm{tw}\,2}(x) = \int_x^1 \frac{dy}{y} \Delta q(y)$$

$$\widehat{h}_L^{\,\mathrm{tw}\,2}(\zeta) = 2\int_0^1 d\alpha\, \alpha\, \widehat{\delta q}(\alpha\zeta), \qquad \Leftrightarrow \qquad h_L^{\mathrm{tw}\,2}(x) = 2x\int_x^1 \frac{dy}{y^2} \delta q(y)$$

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LO factorization, $z^2 \to 0$

Tree order is not so trivial

- Symmetrizing, anti-symmetrizing,...

$$
\begin{aligned}
\mathcal{G}_T(\zeta, z^2) &= \widehat{g}_T^{\mathrm{tw}2}(\zeta) + \widehat{g}_T^{\mathrm{tw}3}(\zeta) + O(z^2) \\
\mathcal{H}_L(\zeta, z^2) &= \widehat{h}_L^{\mathrm{tw}2}(\zeta) + \widehat{h}_L^{\mathrm{tw}3}(\zeta) + O(z^2) \\
\mathcal{E}(\zeta, z^2) &= \Sigma_q + \widehat{e}_{\mathrm{nl}}(\zeta) + O(z^2)
\end{aligned}
$$

- Twist-3 parts ($\bar{\alpha} = 1 - \alpha$)

$$
\begin{aligned}
\widehat{g}_T^{\mathrm{tw}3}(\zeta) &= 2\zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \beta \widehat{S}^-(\zeta, \bar{\beta}\zeta, \bar{\alpha}\zeta) \\
\widehat{h}_L^{\mathrm{tw}3}(\zeta) &= \zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \alpha(2\beta - \alpha) \widehat{H}(\alpha\zeta, \beta\zeta, 0) \\
\widehat{e}_{\mathrm{nl}}(\zeta) &= \zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \widehat{E}(\alpha\zeta, \beta\zeta, 0)
\end{aligned}
$$

LO factorization, $z^2 \to 0$

Tree order is not so trivial

- Symmetrizing, anti-symmetrizing,...

$$\begin{aligned} \mathcal{G}_T(\zeta, z^2) &= \widehat{g}_T^{\,\mathrm{tw}2}(\zeta) + \widehat{g}_T^{\,\mathrm{tw}3}(\zeta) + O(z^2) \\ \mathcal{H}_L(\zeta, z^2) &= \widehat{h}_L^{\,\mathrm{tw}2}(\zeta) + \widehat{h}_L^{\,\mathrm{tw}3}(\zeta) + O(z^2) \\ \mathcal{E}(\zeta, z^2) &= \Sigma_q + \widehat{e}_{\mathrm{nl}}(\zeta) + O(z^2) \end{aligned}$$

- Distribution $\mathcal{E}$ has a local contribution (nuclear-sigma term)

$$2M\Sigma_q = \langle p|\bar{q}(0)q(0)|p\rangle \quad \Rightarrow \qquad \delta(x)\Sigma_q$$

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Why position space is convenient

$$\widehat{f}(\zeta) = \int dx e^{-i\zeta x} f(x), \qquad \widehat{f}(\zeta_1,\zeta_2,\zeta_3) = \int [dx] e^{-i(x_1\zeta + x_2\zeta_2 + x_3\zeta_3)} f(x_1,x_2,x_3)$$

Twist-2 distributions → 1 term in 2 regions

$$\int_0^1 d\alpha \widehat{\Delta q}(\alpha\zeta) \quad \Longleftrightarrow \quad \int dy \int_0^1 d\alpha \delta(x - y\alpha)\Delta q(y) = \theta(x)\int_x^1 \frac{dy}{y}\Delta q(y) - \theta(-x)\int_{-1}^x \frac{dy}{y}\Delta q(y)$$

Twist-3 distributions → 3 terms in 6 regions

$$2\zeta^2 \int_0^1 d\alpha \int_\alpha^1 d\beta \bar{\beta} \widehat{S}^-(\zeta, \beta\zeta, \alpha\zeta) \quad \Longleftrightarrow$$

$$2\int [dx] \int_0^1 d\alpha \left(\frac{\delta(x+\alpha x_1)}{x_1 x_3} + \frac{\delta(x + x_1 + \alpha x_2)}{x_2 x_3} + \frac{\delta(x + x_1)}{x_1 x_2} \right) S^-(x_1,x_2,x_3)$$

- Similar for $e_{\rm nl}$
- For distribution $h_L^{\rm tw3}$ there are 6 terms in 6 regions!

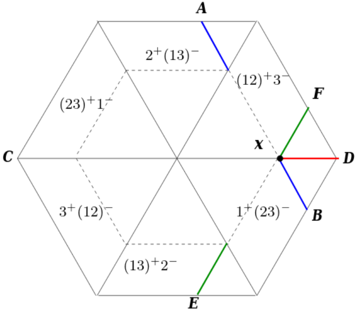
Evolution

tw-2 : $$\frac{d\widehat{\delta q}(\zeta)}{d\ln\mu^2} = a_s \int_0^1 d\alpha \left(\frac{1+\alpha^2}{1-\alpha}\right)_+ \widehat{\delta q}(\alpha\zeta) + a_s^2 ...$$

tw-3 : $$\frac{d}{d\ln\mu^2}\widehat{S}^-(z_1,z_2,z_3) = -a_s[\mathbb{H}\otimes\widehat{S}^-](z_1,z_2,z_3) + a_s^2 ...$$

$$\begin{aligned}
[\mathbb{H}\otimes\widehat{S}](z_1,z_2,z_3) = N_c \int_0^1 d\alpha\Big(&\frac{4}{\alpha}\widehat{S}(z_1,z_2,z_3) - \frac{\bar\alpha}{\alpha}\widehat{S}(z_{12}^\alpha,z_2,z_3) - \frac{\bar\alpha}{\alpha}\widehat{S}(z_1,z_2,z_{32}^\alpha) \\
&- \frac{\bar\alpha^2}{\alpha}\widehat{S}(z_1,z_{21}^\alpha,z_3) - \frac{\bar\alpha^2}{\alpha}\widehat{S}(z_1,z_{23}^\alpha,z_3) - 2\int_0^{\bar\alpha} d\beta\,\bar\beta\widehat{S}(z_1,z_{23}^\beta,z_{32}^\alpha)\Big) \\
&+ \frac{1}{N_c}\int_0^1 d\alpha\Big(\frac{2}{\alpha}\widehat{S}(z_1,z_2,z_3) - \frac{\bar\alpha}{\alpha}\widehat{S}(z_{13}^\alpha,z_2,z_3) - \frac{\bar\alpha}{\alpha}\widehat{S}(z_1,z_2,z_{31}^\alpha) \\
&+ \bar\alpha\,\widehat{S}(z_2,z_{12}^\alpha,z_3) - \int_0^{\bar\alpha} d\beta\,\widehat{S}(z_{13}^\alpha,z_2,z_{31}^\beta) + 2\int_{\bar\alpha}^1 d\beta\,\bar\beta\,\widehat{S}(z_1,z_{23}^\beta,z_{32}^\alpha)\Big) \\
&- 3C_F\widehat{S}(z_1,z_2,z_3).
\end{aligned}$$

$$\boxed{z_{ij}^\alpha = z_i\bar\alpha + z_j\alpha}$$



- For all twist-3 LO evolution kernels see e.g. [Braun, Manashov, Pirnay, 09]
- For relevant LO evolution kernels see appendices

Evolution for $g_T^{\mathbf{tw3}}$

$$\int_0^1 d\alpha \int_\alpha^1 d\beta\, \bar\beta\, [\mathbb{H}\otimes \widehat{S}](\zeta,\beta\zeta,\alpha\zeta) \quad = \quad \int_0^1 d\alpha \int_\alpha^1 d\beta \Big[N_c \bar\beta (\ln\bar\alpha - 2\ln\alpha + 1)$$

$$- \frac{1}{N_c}\left(\alpha\beta - \frac{\alpha^2}{2} + \bar\beta(2\ln\alpha - \ln\bar\alpha)\right) - 3C_F\bar\beta\Big]\widehat{S}(\zeta,\beta\zeta,\alpha\zeta)$$

$$- \frac{1}{N_c}\int_0^1 d\alpha \int_0^\alpha d\beta \Big[\left(\frac{\beta}{\alpha} - \frac{\beta(2-\beta)}{2}\right)\widehat{S}(\zeta,\beta\zeta,\alpha\zeta) + \beta\frac{2-\beta}{2}\widehat{S}(\bar\alpha\zeta,\bar\beta\zeta,0)\Big].$$

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Evolution for $g_T^{\mathbf{tw3}}$

$$\int_0^1 d\alpha \int_\alpha^1 d\beta\, \bar\beta\, [\mathbb{H}\otimes \widehat{S}](\zeta,\beta\zeta,\alpha\zeta) = \int_0^1 d\alpha \int_\alpha^1 d\beta \Big[\boxed{N_c\bar\beta(\ln\bar\alpha - 2\ln\alpha + 1)}$$
$$- \frac{1}{N_c}\left(\alpha\beta - \frac{\alpha^2}{2} + \bar\beta(2\ln\alpha - \ln\bar\alpha)\right) \boxed{-3C_F\bar\beta}\Big]\widehat{S}(\zeta,\beta\zeta,\alpha\zeta)$$
$$- \frac{1}{N_c}\int_0^1 d\alpha \int_0^\alpha d\beta \Big[\left(\frac{\beta}{\alpha} - \frac{\beta(2-\beta)}{2}\right)\widehat{S}(\zeta,\beta\zeta,\alpha\zeta) + \beta\frac{2-\beta}{2}\widehat{S}(\bar\alpha\zeta,\bar\beta\zeta,0)\Big].$$

Large-N_c

At large-N_c equation is diagonal!

$$\frac{d\widehat{g_T}(\zeta)}{d\ln\mu^2} = a_s N_c \int_0^1 d\alpha \left[\left(\frac{1+\alpha}{1-\alpha}\right)_+ + \delta(\bar\alpha)\right]\widehat{g_T}(\alpha\zeta) + \frac{a_s}{N_c}... + a_s^2...$$

Similar case for e and h_L

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LO

$$\begin{aligned} \mathcal{G}_T(\zeta,z^2) &= \widehat{g}_T^{\text{tw}2}(\zeta)+\widehat{g}_T^{\text{tw}3}(\zeta)+O(z^2)\\ \mathcal{H}_L(\zeta,z^2) &= \widehat{h}_L^{\text{tw}2}(\zeta)+\widehat{h}_L^{\text{tw}3}(\zeta)+O(z^2)\\ \mathcal{E}(\zeta,z^2) &= \Sigma_q+\widehat{e}_{\text{nl}}(\zeta)+O(z^2) \end{aligned}$$

Beyond LO

$$\begin{aligned} \mathcal{G}_T(\zeta,z^2) &= C_g\otimes\widehat{g}_1(\zeta)+C_T\otimes\widehat{S^-}(\zeta)+O(z^2)\\ \mathcal{H}_L(\zeta,z^2) &= C_h\otimes\widehat{h}_1(\zeta)+C_L\otimes\widehat{H}^{\text{tw}3}(\zeta)+O(z^2)\\ \mathcal{E}(\zeta,z^2) &= C_\Sigma\otimes\Sigma_q+C_S\otimes\widehat{E}(\zeta)+O(z^2) \end{aligned}$$

- Beyond LO the formulas involve more general terms then combinations g_T, h_L and e_{nl}
- Factorization theorem is “automatically proven” due to the existence of OPE
- We have computed all coefficient functions at NLO using the background field method

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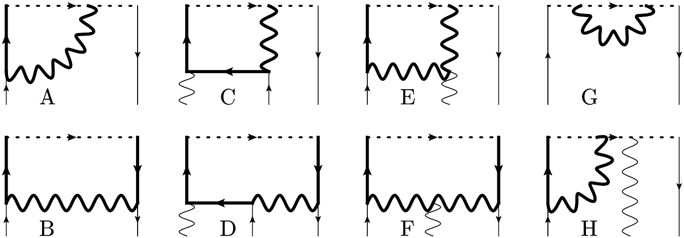
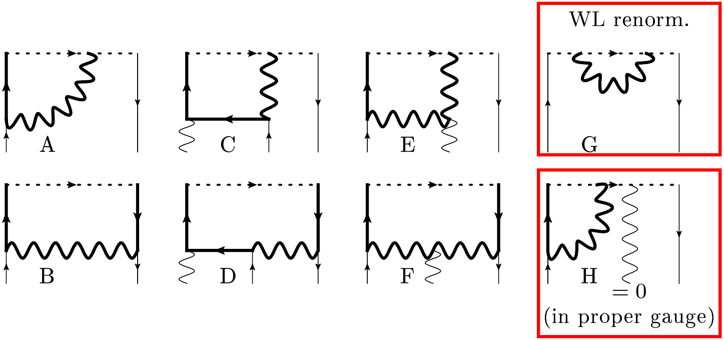
Structure of NLO computation



+mirror diagrams

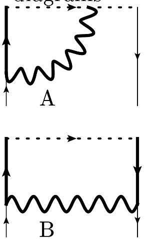
Structure of NLO computation



+mirror diagrams

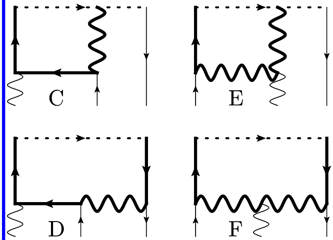
Structure of NLO computation

2-point diagrams



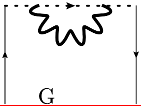
up to 1 derivative
apply EOMs
twist-2 and twist-3

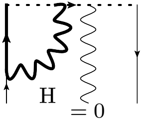
3-point diagrams



→ twist-3 only
straightforward

WL renorm.





(in proper gauge)

+mirror diagrams

NLO computation

- NLO Twist-3 computations are involved
  - No “standard” techniques
  - No dedicated numerical packages
- We made computation in 2+ independent ways:
  - V.Braun: background-field propagator + Schwinger gauge + position space
  - AV: background-field vertices + axial-gauge + position space
  - Yao Ji: background-field vertices + axial-gauge + momentum space
- Checks
  - Cancellation of gauge dependent terms
  - Final expressions for different computations coincides! (very strong check)
  - Agreement with known twist-3 evolution (non-trivial check)
  - Twist-2 parts coincides with literature
- We computed ITDs (it is simpler)
  - Fourier transform ITDs to pPDFs (all known parts coincides!)
  - Double-Fourier transform pPDFs to qPDFs (all known parts coincides!)
- The expressions for $\mathcal{G}_T$ [2103.12105]
- The expressions for $\mathcal{E}$ and $\mathcal{H}_L$ [2108.03065]

Results are lengthy. I am not presenting them explicitly here. See publication

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Presentation of the result
3-point VS. 2-point

$$\widehat{g}_T^{\text{tw}3}(\zeta) = 2\zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \beta \widehat{S}^-(\zeta, \bar{\beta}\zeta, \bar{\alpha}\zeta)$$

$$\widehat{h}_L^{\text{tw}3}(\zeta) = \zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \alpha(2\beta - \alpha)\widehat{H}(\alpha\zeta, \beta\zeta, 0)$$

$$\widehat{e}_{\text{nl}}(\zeta) = \zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \widehat{E}(\alpha\zeta, \beta\zeta, 0)$$

3pt distribution

- "True" QFT functions
- Complicated expressions
- Complicated numerics

2pt distribution

- "Fake" QFT functions
- Just like usual PDFs
- Tree order

In the final expression we rewrite 3pt distributions via 2pt distribution,
where it is possible

Use symmetry relations

$$S^+(x_1,x_2,x_3) = +S^+(-x_3,-x_2,-x_1) \qquad S^-(x_1,x_2,x_3) = -S^-(-x_3,-x_2,-x_1)$$

$$H(x_1,x_2,x_3) = -H(-x_3,-x_2,-x_1) \qquad E(x_1,x_2,x_3) = +E(-x_3,-x_2,-x_1)$$

Examples:

$$\zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \alpha\bar{\alpha}(2\beta-\alpha)\, \widehat{H}(\alpha\zeta,\beta\zeta,0) = \int_0^1 d\alpha \alpha^2 \widehat{h}_L^{\text{tw3}}(\alpha\zeta)$$

$$\zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \ln\bar{\alpha}\, \widehat{E}(\alpha\zeta,\beta\zeta,0) = \int_0^1 d\alpha \left(\frac{1}{1-\alpha}\right)_+ \widehat{e}_{nl}(\alpha\zeta)$$

$$\zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta\, \ln\beta\, \widehat{E}(\alpha\zeta,\beta\zeta,0) = \text{cannot be rewritten as 2pt}$$

$$\zeta^2 \int_0^1 d\alpha \int_\alpha^1 d\beta\, \ln\bar{\alpha}\, \widehat{E}(\alpha\zeta,\beta\zeta,0) = \text{cannot be rewritten as 2pt}$$

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The factorization theorem has the form

$$\widehat{\mathcal{G}}_T(\zeta) = \underbrace{\int_0^1 d\alpha \mathbf{C}_1(\alpha)\widehat{\Delta q}(\alpha\zeta)}_{\text{tw-2 part}} + \underbrace{\int_0^1 d\alpha \mathbf{C}_{T,2\text{pt}}(\alpha)\widehat{g}_T^{\text{tw3}}(\alpha\zeta)}_{\text{tw-3 2pt part}} + \underbrace{\zeta^2 \mathbf{C}_{L,3\text{pt}} \otimes \widehat{S}^-}_{\text{tw-3 3pt part}} + O(z^2)$$

- Same form for $\mathcal{H}_L$
- Similar form for $\mathcal{E}$ with tw-2 part $\to C_\Sigma \Sigma_q$
- Coefficient functions depends on

$$L_z = \ln\left(\frac{-z^2\mu^2}{4e^{-2\gamma_E}}\right)$$

- 3-point part contains "irreducible" 3-point terms which cannot be presented via 2-pt

Example of expression

Twist-3 part

$$\mathcal{G}_T^{\text{tw3}}(\zeta,z^2)=\widehat{g}_T^{\text{tw3}}(\zeta;\mu)+a_s\mathbf{C}_{2\text{pt}}^{(1)}\otimes\widehat{g}_T^{\text{tw3}}+2\zeta^2 a_s\mathbf{C}_{3\text{pt}}^{(1)}\otimes\widehat{S}^- \tag{4.31}$$

with

$$\mathbf{C}_{2\text{pt}}^{(1)}\otimes\widehat{g}_T^{\text{tw3}}=\int_0^1 d\alpha\left[\mathbf{C}_T^{(1)}(\alpha,\mathrm{L}_z;\mu)+N_c\Big(\mathrm{L}_z\left(\delta(\bar\alpha)-\alpha\right)+\alpha+2\delta(\bar\alpha)\Big)\right]\widehat{g}_T^{\text{tw3}}(\alpha\zeta;\mu)\,, \tag{4.32}$$

$$\begin{aligned}\mathbf{C}_{3\text{pt}}^{(1)}\otimes\widehat{S}^- =& -\mathrm{L}_z\,\mathrm{P}_{\text{tw3}}\otimes\widehat{S}^- +\int_0^1 d\alpha\Big\{\int_\alpha^1 d\beta\left(2N_c\ln\beta+\frac{1}{N_c}\frac{\alpha^2}{2}\right)\widehat{S}^-(\zeta,\beta\zeta,\alpha\zeta)\\ &+\frac{1}{N_c}\int_0^\alpha d\beta\left[\frac{\beta^2}{2}\widehat{S}^-(\bar\alpha\zeta,\bar\beta\zeta,0)-\Big(\frac{\beta(2+\beta)}{2}-\frac{2\beta}{\alpha}(1+\ln\alpha)\Big)\widehat{S}^-(\zeta,\beta\zeta,\alpha\zeta)\right]\Big\},\end{aligned} \tag{4.33}$$

where the logarithmic part is given by

$$\begin{aligned}\mathrm{P}_{\text{tw3}}\otimes\widehat{S}^- =& \frac{1}{N_c}\int_0^1 d\alpha\Big\{\int_\alpha^1 d\beta\frac{\alpha(\alpha-2)}{2}\widehat{S}^-(\zeta,\beta\zeta,\alpha\zeta)\\ &+\int_0^\alpha d\beta\left[\frac{\beta(\beta-2)}{2}\widehat{S}^-(\bar\alpha\zeta,\bar\beta\zeta,0)+\Big(\frac{\beta(2-\beta)}{2}-\frac{\beta}{\alpha}\Big)\widehat{S}^-(\zeta,\beta\zeta,\alpha\zeta)\right]\Big\}.\end{aligned} \tag{4.34}$$

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Example of expression

Twist-3 part

$$\mathcal{G}_T^{\text{tw3}}(\zeta, z^2) = \widehat{g}_T^{\text{tw3}}(\zeta;\mu) + a_s \mathbf{C}_{2\text{pt}}^{(1)} \otimes \widehat{g}_T^{\text{tw3}} + 2\zeta^2 a_s \mathbf{C}_{3\text{pt}}^{(1)} \otimes \widehat{S}^- \quad (4.31)$$

with

$$\mathbf{C}_{2\text{pt}}^{(1)} \otimes \widehat{g}_T^{\text{tw3}} = \int_0^1 d\alpha \left[\mathbf{C}_T^{(1)}(\alpha, \mathrm{L}_z;\mu) + N_c\Big(\mathrm{L}_z\left(\delta(\bar\alpha) - \alpha\right) + \alpha + 2\delta(\bar\alpha)\Big)\right] \widehat{g}_T^{\text{tw3}}(\alpha\zeta;\mu), \quad (4.32)$$

$$\mathbf{C}_{3\text{pt}}^{(1)} \otimes \widehat{S}^- = -\mathrm{L}_z \mathrm{P}_{\text{tw3}} \otimes \widehat{S}^- + \int_0^1 d\alpha \Big\{ \int_\alpha^1 d\beta \left(2N_c \ln\beta + \frac{1}{N_c}\frac{\alpha^2}{2}\right)\widehat{S}^-(\zeta, \beta\zeta, \alpha\zeta)$$
$$+ \frac{1}{N_c}\int_0^\alpha d\beta \left[\frac{\beta^2}{2}\widehat{S}^-(\bar\alpha\zeta, \bar\beta\zeta, 0) - \left(\frac{\beta(2+\beta)}{2} - \frac{2\beta}{\alpha}(1+\ln\alpha)\right)\widehat{S}^-(\zeta, \beta\zeta, \alpha\zeta)\right]\Big\}, \quad (4.33)$$

non-2pt contribution at large N_c

where the logarithmic part is given by

$$\mathrm{P}_{\text{tw3}} \otimes \widehat{S}^- = \frac{1}{N_c}\int_0^1 d\alpha \Big\{ \int_\alpha^1 d\beta \frac{\alpha(\alpha-2)}{2}\widehat{S}^-(\zeta, \beta\zeta, \alpha\zeta)$$
$$+ \int_0^\alpha d\beta \left[\frac{\beta(\beta-2)}{2}\widehat{S}^-(\bar\alpha\zeta, \bar\beta\zeta, 0) + \left(\frac{\beta(2-\beta)}{2} - \frac{\beta}{\alpha}\right)\widehat{S}^-(\zeta, \beta\zeta, \alpha\zeta)\right]\Big\}. \quad (4.34)$$

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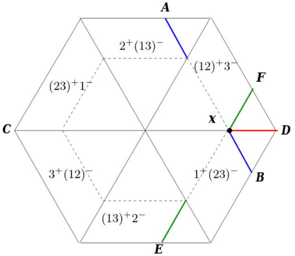
From ITDs to pPDFs

$$\mathfrak{g}_T(x,z^2)=\int\frac{d\zeta}{2\pi}e^{-ix\zeta}\mathcal{G}_T(\zeta,z^2)$$

$$\mathfrak{g}_T(x,z^2)=g_T(x)+a_s\int_{|x|}^1\frac{d\alpha}{\alpha}\left(\mathfrak{C}_T(\alpha,\mathrm{L}_z)g_T^{\mathrm{tw}\,2}\left(\frac{x}{\alpha}\right)+\mathfrak{C}^{(1)}_{2\mathrm{pt}}(\alpha,\mathrm{L}_z)g_T^{\mathrm{tw}3}\left(\frac{x}{\alpha}\right)\right)+2a_s\mathfrak{C}^{(1)}_{3\mathrm{pt}}\otimes S^-$$

2pt pPDF coeff. functions = 2pt qPDF coeff. function

$$\begin{aligned}\mathfrak{C}^{(1)}_{3\mathrm{pt}}\otimes S^- = &-\mathrm{L}_z\mathfrak{P}_{\mathrm{tw3}}\otimes S^-\\ &+\int[dx]\int_0^1 d\alpha\left\{\frac{-2N_c}{1-\alpha}\left(\frac{\delta(x+\alpha x_1)}{x_1x_3}+\frac{\delta(x-x_3-\alpha x_2)}{x_2x_3}+\frac{\delta(x+x_1)}{x_1x_2}\right)\right.\\ &+\frac{1}{N_c}\left[-\frac{2}{(1-\alpha)_+}\frac{\delta(x+\alpha x_1)}{x_1x_2}+\frac{2(1+\ln\bar\alpha)}{1-\alpha}\frac{\delta(x+\alpha x_1)-\delta(x+x_1+\bar\alpha x_3)}{x_2^2}\right.\\ &-\bar\alpha\left(\frac{\delta(x-\alpha x_3)}{x_1x_2}+\frac{\delta(x-\alpha x_2)}{x_2x_3}-\frac{\delta(x+\alpha x_2)}{x_1x_2}-\frac{\delta(x+\alpha x_1)}{x_1x_2}+\frac{\delta(x+\alpha x_1)}{x_1x_3}\right)\\ &\left.\left.-\left(\frac{\delta(x-\alpha x_2)}{x_2x_3}+\frac{\delta(x+\alpha x_1)}{x_1x_3}+2\frac{\delta(x+x_1)}{x_1x_2}\right)\right]\right\}S^-(x_1,x_2,x_3)\,,\end{aligned}\qquad(5.9)$$



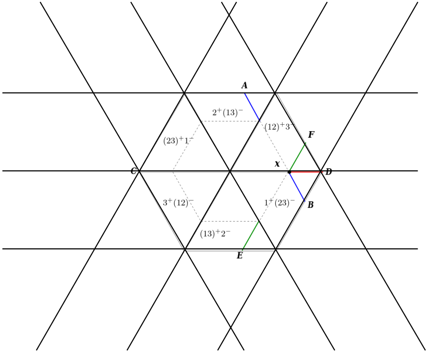
From pPDFs to qPDFs

$$\mathfrak{g}_i(x, p_v) = \int \frac{d\zeta}{2\pi} \int_{-1}^{1} dy \; e^{i(y-x)\zeta} \mathfrak{g}_i\left(y, \frac{\zeta^2}{p_v^2}\right)$$

Long expressions!
see (5.11)-(5.21) in
[2103.12105]
see (4.42)-(4.50) in
[2103.12105]

30 domains!
12 singular points!

qPDF representation is just
inefficient terminology for higher-twists



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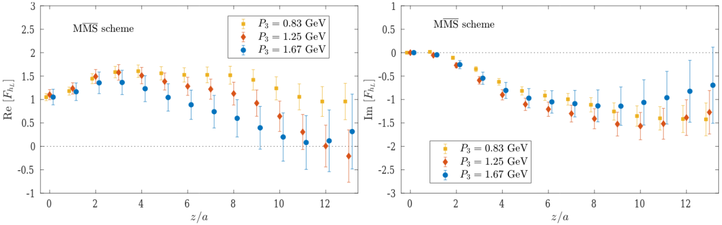
How to extract twist-3 distributions from lattice?

Problem: Given measurement of qITD, pPDF, qPDF extract (some information about) twist-3 PDF

What is the problem?

- Lattice measurements, already some [S.Bhattachary, et al,2004.04130], [S.Bhattachary, et al,2107.02574]



- **How to clean away "uninteresting" twist-2 part?**
- No such problem for distribution $\mathcal{E}$ (but Σ-term)
- For distributions $\mathcal{H}_L$ and $\mathcal{G}_T$ situations are different

Let me start from h_L

$$\langle p,s|\bar{q}(z)i\sigma^{\mu z}\gamma^5 q(0)|p,s\rangle = 2s_T^\mu \zeta \mathcal{H}_1(\zeta,z^2) - (sz)\left(z^\mu - p^\mu \frac{z^2}{\zeta}\right) M\mathcal{H}_L(\zeta,z^2)$$

not-interesting $\mathcal{H}_L^{\text{tw}2} + \mathcal{H}_L^{\text{tw}3}$ interesting

In fact, there is only one twist-2 "structure function" $\sim 2S^\mu\zeta$
It is decribed by $\langle p|[\bar{q}(z)i\sigma^{\mu z}\gamma^5 q(0)]^{\text{tw}2}|p\rangle$
Thus, $\mathcal{H}_1$ and $\mathcal{H}_L^{\text{tw}2}$ are just different projections of it.
They are connected by **exact relation**

$$\mathcal{H}_L^{\text{tw}2}(\zeta,z^2) = 2\int_0^1 d\alpha\,\alpha\,\mathcal{H}_1(\alpha\zeta,z^2)$$

We called it Jaffe-Ji relation (JJ) since similar relation has been derived in [Jaffe,Ji,91]

$$\langle p,s|\bar{q}(z) i\sigma^{\mu z}\gamma^5 q(0)|p,s\rangle = 2s_T^\mu \zeta \mathcal{H}_1(\zeta,z^2) - (sz)\left(z^\mu - p^\mu \frac{z^2}{\zeta}\right) M\mathcal{H}_L(\zeta,z^2)$$

not-interesting $\mathcal{H}_L^{\text{tw2}}$ + $\mathcal{H}_L^{\text{tw3}}$ interesting

In fact, there is only one twist-2 "structure function" $\sim 2S^\mu\zeta$

It is decribed by $\langle p|[\bar{q}(z) i\sigma^{\mu z}\gamma^5 q(0)]^{\text{tw2}}|p\rangle$

Thus, $\mathcal{H}_1$ and $\mathcal{H}_L^{\text{tw2}}$ are just different projections of it.

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$$\mathcal{H}_L^{\text{tw2}}(\zeta,z^2) = 2\int_0^1 d\alpha\, \alpha\, \mathcal{H}_1(\alpha\zeta, z^2)$$

We called it Jaffe-Ji relation (JJ) since similar relation has been derived in [Jaffe,Ji,91]

Chicking JJ relation at NLO

$$\mathcal{H}_1(\zeta,z^2) = \widehat{\delta q}(\zeta) + a_s \int_0^1 d\alpha \mathbf{C}_1^{(1)}(\alpha)\widehat{\delta q}(\alpha\zeta) + O(z^2)$$

$$\widehat{h}_L^{\text{tw2}}(\zeta) = 2\int_0^1 d\alpha\, \alpha\, \widehat{\delta q}(\alpha\zeta)$$

same!

$$\mathcal{H}_L(\zeta,z^2) = \widehat{h}_L^{\text{tw2}}(\zeta) + a_s \int_0^1 d\alpha \mathbf{C}_1^{(1)}(\alpha)\widehat{h}_L^{\text{tw2}}(\alpha\zeta) + \text{twist-3} + O(z^2)$$

$$\langle p,s|\bar q(z)i\sigma^{\mu z}\gamma^5 q(0)|p,s\rangle = 2s_T^\mu \zeta \mathcal{H}_1(\zeta,z^2) - (sz)\left(z^\mu - p^\mu \frac{z^2}{\zeta}\right) M\mathcal{H}_L(\zeta,z^2)$$

$$\mathcal{H}_L = \mathcal{H}_L^{\text{tw}2} + \mathcal{H}_L^{\text{tw}3}$$

not-interesting: $\mathcal{H}_L^{\text{tw}2}$; interesting: $\mathcal{H}_L^{\text{tw}3}$

In fact, there is only one twist-2 "structure function" $\sim 2S^\mu\zeta$

It is decribed by $\langle p|[\bar q(z)i\sigma^{\mu z}\gamma^5 q(0)]^{\text{tw}2}|p\rangle$

Thus, $\mathcal{H}_1$ and $\mathcal{H}_L^{\text{tw}2}$ are just different projections of it.

They are connected by **exact relation**

$$\mathcal{H}_L^{\text{tw}2}(\zeta,z^2) = 2\int_0^1 d\alpha\,\alpha\,\mathcal{H}_1(\alpha\zeta,z^2)$$

We called it Jaffe-Ji relation (JJ) since similar relation has been derived in [Jaffe,Ji,91]

$$\mathcal{H}_L^{\text{tw}3}(\zeta,z^2) = \mathcal{H}_L(\zeta,z^2) - 2\int_0^1 d\alpha\,\alpha\,\mathcal{H}_1(\alpha\zeta,z^2)$$

twist-3 part can be extracted using the same lattice simulation!

JJ-relation is exact (at all orders of PT)

It also exact for pPDFs

$$\mathfrak{h}_L^{\text{tw3}}(x,z^2)=\mathfrak{h}_L(x,z^2)-2x\int_x^1\frac{dy}{y^2}\mathfrak{h}_1(y,z^2)$$

JJ-relation is violated for qPDFs

$$\mathtt{h}_L(x,p_v)-2\int_{|x|}^1 dy\,\mathtt{h}_1\left(\frac{x}{y},p_v\right)=\mathtt{h}_L^{\text{tw3}}(x,p_v)$$
$$+8a_sC_F\int_{|x|}^1 dy\left(2\ln y\ln\bar{y}-\ln^2 y+2\text{Li}_2(\bar{y})\right)\delta q\left(\frac{x}{y}\right)+O(a_s^2)$$

- Fourier transformation for qPDF changes the factorization scale z^2
- Terms Fourier of terms $\sim\ln(z^2)$ violate JJ-relation
- In principle can be subtracted perturbatively

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What about axial case?

$$\langle p,s|\bar q(z)\gamma^\mu\gamma^5 q(0)|p,s\rangle = 2p^\mu \frac{(sz)}{\zeta} M\mathcal{G}_1(\zeta,z^2) + s_T^\mu M\mathcal{G}_T(\zeta,z^2) + \ldots$$

not-interesting $\mathcal{G}_T^{\text{tw}\,2} + \mathcal{G}_T^{\text{tw}\,3}$ interesting

There is only one twist-2 "structure function" $\sim 2S^\mu M$

It is decribed by $\langle p|[\bar q(z)i\gamma^\mu q(0)]^{\text{tw}\,2}|p\rangle$

Thus, $\mathcal{G}_1$ and $\mathcal{G}_T^{\text{tw}\,2}$ are just different projections of it.

They are connected by **exact relation**

$$\mathcal{G}_T^{\text{tw}\,2}(\zeta,z^2) = \int_0^1 d\alpha\, \mathcal{G}_1(\alpha\zeta,z^2)$$

It is called Wandzura-Wilczek relation (WW) since similar relation has been derived for DIS structure functions in [WW,77]

What about axial case?

$$\langle p,s|\bar q(z)\gamma^\mu\gamma^5 q(0)|p,s\rangle = 2p^\mu \frac{(sz)}{\zeta} M\mathcal{G}_1(\zeta,z^2) + s_T^\mu M\mathcal{G}_T(\zeta,z^2)+\dots$$

not-interesting $\mathcal{G}_T^{\text{tw}2} + \mathcal{G}_T^{\text{tw}3}$ interesting

There is only one twist-2 "structure function" $\sim 2S^\mu M$

It is decribed by $\langle p|[\bar q(z)i\gamma^\mu q(0)]^{\text{tw}2}|p\rangle$

Thus, $\mathcal{G}_1$ and $\mathcal{G}_T^{\text{tw}2}$ are just different projections of it.

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It is called Wandzura-Wilczek relation (WW) since similar relation has been derived for DIS structure functions in [WW,77]

Chicking WW relation at NLO

$$\mathcal{G}_1(\zeta,z^2) = \widehat{\Delta q}(\zeta) + a_s\int_0^1 d\alpha \mathbf{C}_1^{(1)}(\alpha)\widehat{\Delta q}(\alpha\zeta) + O(z^2)$$

$$\widehat{g}_T^{\text{tw}2}(\zeta) = \int_0^1 d\alpha\,\widehat{\Delta q}(\alpha\zeta)$$

different!

$$\mathcal{G}_T(\zeta,z^2) = \widehat{g}_T^{\text{tw}2}(\zeta) + a_s\int_0^1 d\alpha \mathbf{C}_T^{(1)}(\alpha)\widehat{g}_T^{\text{tw}2}(\alpha\zeta) + \text{twist-3} + O(z^2)$$

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What about axial case?

$$\langle p,s|\bar q(z)\gamma^\mu\gamma^5 q(0)|p,s\rangle = 2p^\mu \frac{(sz)}{\zeta} M\mathcal{G}_1(\zeta,z^2) + s_T^\mu M\mathcal{G}_T(\zeta,z^2)+...$$

not-interesting $\mathcal{G}_T^{\text{tw}2} + \mathcal{G}_T^{\text{tw}3}$ interesting

There is only one twist-2 "structure function" $\sim 2S^\mu M$
It is decribed by $\langle p|[\bar q(z)i\gamma^\mu q(0)]^{\text{tw}2}|p\rangle$
Thus, $\mathcal{G}_1$ and $\mathcal{G}_T^{\text{tw}2}$ are just different projections of it.
They are connected by **exact relation**

$$\mathcal{G}_T^{\text{tw}2}(\zeta,z^2) = \int_0^1 d\alpha\, \mathcal{G}_1(\alpha\zeta,z^2)$$

It is called Wandzura-Wilczek relation (WW) since similar relation has been derived for DIS structure functions in [WW,77]

In the case of quasi-distribution there is an extra structure function $\sim \frac{z^\mu z^\nu}{z^2}$
It is zero at tree-order but appears at NLO (see e.g. [Radyushkin,17; Braun,et al,18; Izubuchi, et al,18])
Thus, there is no exact WW relation for $\mathcal{G}_T$ case

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No exact WW-relation

No WW-relation for qITDs and pPDFs

$$\mathfrak{g}_T(x,z^2) - \int_{|x|}^1 \frac{dy}{y}\mathfrak{g}_1(y,z^2) = 4a_sC_F\int_{|x|}^1 \frac{dy}{y}(\bar{y}+\ln y)\Delta q\Big(\frac{x}{y}\Big) + \text{twist-three} + O(a_s^2)$$

No WW-relation for qITDs and pPDFs

$$\mathfrak{g}_T(x,p_v) - 2\int_{|x|}^1 dy\,\mathfrak{g}_1\Big(\frac{x}{y},p_v\Big) =$$

$$8a_sC_F\int_{|x|}^1 dy\Big(\ln y\ln\bar{y} - \frac{\ln^2 y}{4} + \text{Li}_2(\bar{y})\Big)\Delta q\Big(\frac{x}{y}\Big) + \text{twist-three} + O(a_s^2)$$

- In principle can be subtracted perturbatively
- Numerical estimate shows that violation is large $\sim 100\%$ of the twist-3 part.
- No WW relation $\Rightarrow$ no Burkhardt-Cottingham sum rules

The factorization theorem for twist-3 quasi-distributions is derived!

Theory side

- All (simplest) twist-3 qITDs (pPDFs, qPDFs) are considered $\sim g_T(x),\ h_L(x),\ e(x)$
- LO and NLO expressions for qITDs, pPDFs, qPDFs are derived
- Nothing principally new $\Rightarrow$ routine twist-3 computation

Practical side

- Distribution $\mathcal{E} \sim e(x)$ is pure twist-3
- Distribution $\mathcal{H}_L \sim h_L(x)$ has twist-2 part
  - It can be purified by means of exact JJ-relation for qITDs and pPDFs
  - qPDF case violates JJ relation (but it could be improved perturbatively)
- Distribution $\mathcal{G}_T \sim g_T(x)$ has twist-2 part
  - WW-relation is violated by a "hidden" tensor structure $\sim z^\mu z^\nu / z^2$

I believe, that lattice simulation will provide an important input in tw3-physics. But it could be not that simple.

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Backup slides

IMPORTANT NOTE: no light-cone

$z^2 \neq 0$

it is only a part of expression

see e.g. (5.10) [Balitsky & Braun,89]

$$[\bar{q}(z)[z,0]\gamma^\mu q(0)]^{\mathrm{tw}-2} = \int_0^1 d\alpha \frac{\partial}{\partial z_\mu} \bar{q}(\alpha z)[\alpha z, 0]\not{z} q(0) + O(z^2)$$

$$\langle p|[\bar{q}(z)[z,0]\gamma^\mu q(0)]^{\mathrm{tw}-2}|p\rangle = 2p^\mu \int_{-1}^1 dx e^{ix(pz)} f_1(x)$$

twist-2 PDF

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$$[\bar{q}(z)[z,0]\gamma^\mu q(0)]^{\text{tw}-2} = \int_0^1 d\alpha \frac{\partial}{\partial z_\mu} \bar{q}(\alpha z)[\alpha z, 0] \not{z} q(0) + O(z^2)$$

$$\langle p|[\bar{q}(z)[z,0]\gamma^\mu q(0)]^{\text{tw}-2}|p\rangle = 2p^\mu \int_{-1}^1 dx e^{ix(pz)} f_1(x)$$

twist-2 PDF

integration by parts

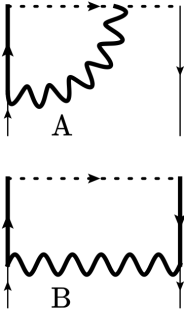
$z^2 = 0$

$$[\bar{q}(z)[z,0]\gamma^\mu q(0)]^{\text{tw}-2} = \bar{n}^\mu \bar{q}(z)[z,0]\gamma^+ q(0)$$

$$\langle p|\bar{q}(z)[z,0]\gamma^+ q(0)\Big|_{z^2=0}||p\rangle = 2p^+ \int_{-1}^1 dx e^{ix(pz)} f_1(x)$$

same twist-2 PDF

Light-cone limit is very helpful! (but not necessary)



Important points

$$
\begin{aligned}
\mathbf{B} \;=\; & a_s C_F \Gamma(-\epsilon)\mathbf{Z}^{\epsilon}\int_0^1 [d\alpha d\beta d\gamma]\bar{q}(z_{12}^{\alpha})\Big\{\frac{\gamma^{\mu}\gamma^{\nu}\Gamma\gamma^{\nu}\gamma^{\mu}}{2}+\epsilon\frac{\gamma^{\mu}\not v\Gamma\not v\gamma^{\mu}}{v^2} \\
& +\frac{z_{12}}{2}\Big[-\alpha\gamma^{\mu}\not v\Gamma\overleftarrow{\not\partial}\gamma^{\mu}-\bar{\beta}\gamma^{\mu}\not v\Gamma\overrightarrow{\not\partial}\gamma^{\mu}+\bar{\alpha}\gamma^{\mu}\overleftarrow{\not\partial}\Gamma\not v\gamma^{\mu}+\beta\gamma^{\mu}\overrightarrow{\not\partial}\Gamma\not v\gamma^{\mu}\Big]\Big\}q(z_{21}^{\beta}) \\
& \hfill +z^2...
\end{aligned}
$$

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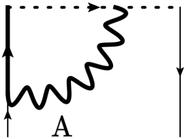

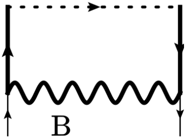

Important points

$$\mathbf{B} = a_s C_F \Gamma(-\epsilon)\mathbf{Z}^\epsilon \int_0^1 [d\alpha d\beta d\gamma] \bar{q}(z_{12}^\alpha)\Big\{ \boxed{\frac{\gamma^\mu\gamma^\nu\Gamma\gamma^\nu\gamma^\mu}{2} + \epsilon \frac{\gamma^\mu \not v \Gamma \not v \gamma^\mu}{v^2}} + \frac{z_{12}}{2}\Big[-\alpha\gamma^\mu \not v \Gamma \overleftarrow{\not\partial}\gamma^\mu - \bar\beta\gamma^\mu \not v\Gamma\overrightarrow{\not\partial}\gamma^\mu + \bar\alpha\gamma^\mu\overleftarrow{\not\partial}\Gamma\not v\gamma^\mu + \beta\gamma^\mu\overrightarrow{\not\partial}\Gamma\not v\gamma^\mu\Big]\Big\} q(z_{21}^\beta) + z^2 ...$$

Mixes structure functions!

$$\bar{q}\gamma^\mu q \sim C_\parallel \otimes \bar{q}\gamma^\mu q + \frac{z^\mu z^\nu}{z^2} C_\perp \otimes \bar{q}\gamma_\nu q$$

$$C_\parallel = O(1), \qquad C_\perp = O(\alpha_s)$$

known for long time e.g. [Radyushkin,17; Braun,et al,18; Izubuchi, et al,18]

Leads to violation of WW-relation for structure functions

$$\mathcal{G}_T^{\text{tw2}}(\zeta, z^2) - \int_0^1 d\alpha\, \mathcal{G}_1(\alpha\zeta, \alpha^2 z^2) =$$

$$= 8a_s C_F \int_0^1 d\alpha \left(\text{Li}_2(\bar\alpha) + \ln\bar\alpha\ln\alpha - \frac{\ln^2\alpha}{4}\right)\widehat{\Delta q}\,(\alpha\zeta) + \mathcal{O}(a_s^2)$$

no such problem for h_L and $e_{\mathbf{nl}}$

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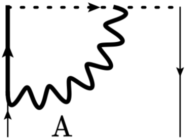
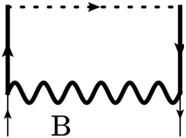
Important points



A



B

$$
\begin{aligned}
\mathbf{B} \;=\; & a_s C_F \Gamma(-\epsilon)\mathbf{Z}^\epsilon \int_0^1 [d\alpha d\beta d\gamma]\bar{q}(z_{12}^\alpha)\Big\{\frac{\gamma^\mu\gamma^\nu\Gamma\gamma^\nu\gamma^\mu}{2} + \epsilon\frac{\gamma^\mu \not{v}\Gamma\not{v}\gamma^\mu}{v^2} \\
& + \frac{z_{12}}{2}\Big[-\alpha\gamma^\mu\not{v}\Gamma\overleftarrow{\not{\partial}}\gamma^\mu - \bar{\beta}\gamma^\mu\not{v}\Gamma\overrightarrow{\not{\partial}}\gamma^\mu + \bar{\alpha}\gamma^\mu\overleftarrow{\not{\partial}}\Gamma\not{v}\gamma^\mu + \beta\gamma^\mu\overrightarrow{\not{\partial}}\Gamma\not{v}\gamma^\mu\Big]\Big\}q(z_{21}^\beta) \\
& + z^2 \ldots
\end{aligned}
$$

2 point $\to$ 3 point

$\not{\partial} q \to +ig\not{A}q + \text{EOM}$

This term is not gauge-invariant!

Gauge-dependance cancels once summed with 3-point diags.

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