Mellin Moments of the Pion Light-Cone Distribution Amplitude Using the HOPE Method

THE HOPE COLLABORATION

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Outline

Light-Cone Distribution Amplitude

- LCDA $\varphi_\pi(\xi)$ defined via

$$\langle 0|\bar{d}(-z)\gamma_\mu\gamma_5\mathcal{W}[-z,z]u(z)|\pi^+(p)\rangle = ip_\mu f_\pi \int_{-1}^{1} d\xi\, e^{-i\xi p\cdot z}\varphi_\pi(\xi)$$

- Represents amplitude for π transitioning into $q\bar{q}$ pair with momenta $(1+\xi)p/2$, $(1-\xi)p/2$
- Many physical properties depend on φ_π

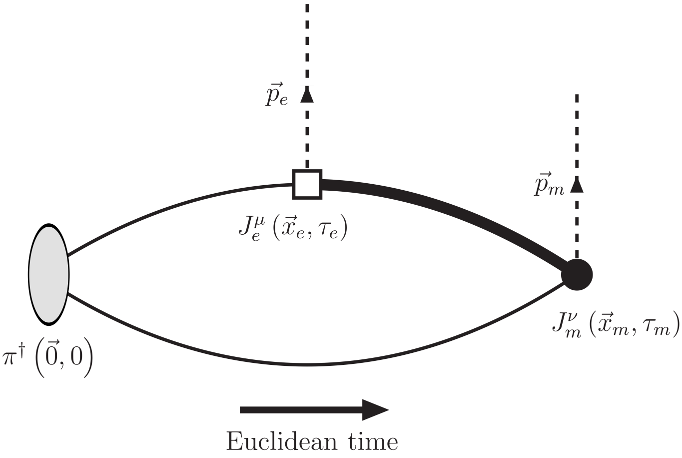
Lattice Determination of LCDA

- Our approach: expand LCDA into Mellin moments

$$\langle \xi^n \rangle = \int_{-1}^{1} d\xi \, \xi^n \varphi_\pi(\xi)$$

  - Computation of $\langle \xi^2 \rangle$
  - Exploratory computation of $\langle \xi^4 \rangle$
- Previous lattice calculations
  - Local matrix elements
  - Light-quark operator product expansion
  - Quasi-PDF and pseudo-PDF

Heavy-Quark Operator Product Expansion (HOPE)



Heavy-Quark Operator Product Expansion (HOPE)

$$V^{\mu\nu}(q,p) = \int d^4x\, e^{iq\cdot x} \langle 0|\mathcal{T}\left[A^\mu(x/2)A^\nu(-x/2)\right]|\pi^+(p)\rangle$$

$$A^\mu = \bar{\Psi}\gamma^\mu\gamma_5\psi + \bar{\psi}\gamma^\mu\gamma_5\Psi$$

- Hadronic tensor can be expanded in terms of moments

$$V^{\mu\nu}(p,q) = \frac{2if_\pi \varepsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma}{\tilde{Q}^2} \sum_{\substack{n=0\\ \text{even}}}^{\infty} \frac{\tilde{\omega}^n}{2^n(n+1)} C_W^{(n)}(\tilde{Q}, m_\Psi, \mu)\langle \xi^n\rangle(\mu) + O\left(\frac{\Lambda_{\text{QCD}}}{\tilde{Q}}\right)$$

with $\tilde{\omega} = 2p\cdot q/\tilde{Q}^2$ and $\tilde{Q}^2 = -q^2 - m_\Psi^2$

- Heavy quark mass m_Ψ suppresses higher-twist effects

Hadronic Tensor

$$V^{\mu\nu}(q,p) = \int d^4x\, e^{iq\cdot x} \left\langle 0 \left| \mathcal{T}\left[A^{\mu}\left(\frac{x}{2}\right) A^{\nu}\left(-\frac{x}{2}\right)\right] \right| \pi^+(p) \right\rangle$$

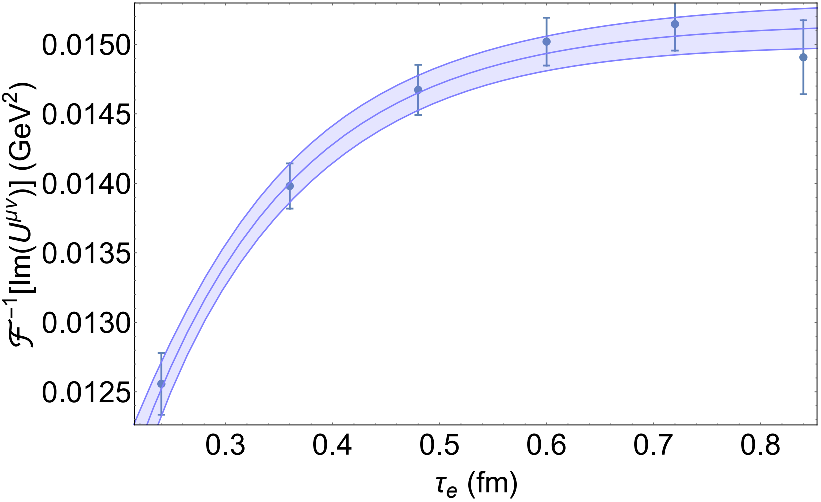
$$\int dq_4 e^{-iq_4\tau} V^{\mu\nu}(q,p) = \int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} \left\langle 0 \left| \mathcal{T}\left[A^{\mu}\left(\frac{\mathbf{x}}{2},\frac{\tau}{2}\right) A^{\nu}\left(-\frac{\mathbf{x}}{2},-\frac{\tau}{2}\right)\right] \right| \pi^+(\mathbf{p}) \right\rangle$$

- Inverse FT of $V^{\mu\nu}$ calculable on lattice in terms of 2-point and 3-point functions

$$C_2(\tau) = \langle \mathcal{O}_\pi(\tau)\mathcal{O}_\pi^\dagger(0)\rangle$$
$$C_3(\tau_e,\tau_m) = \langle A^\mu(\tau_e)A^\nu(\tau_m)\mathcal{O}_\pi^\dagger(0)\rangle$$

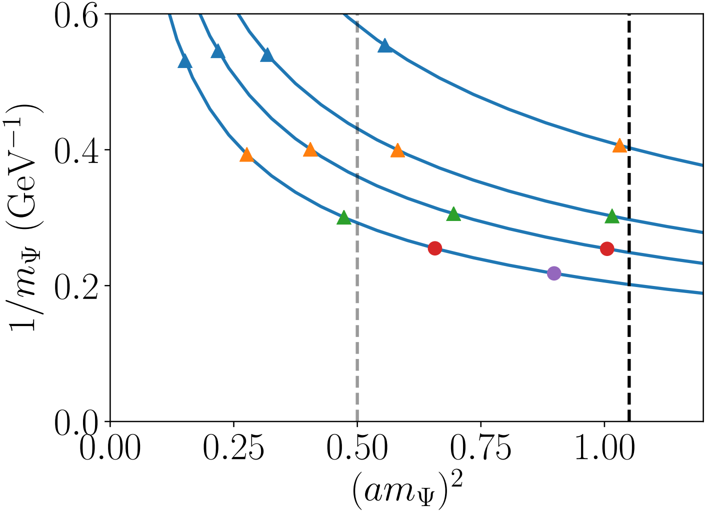
- Isolation of ground state relies on sufficiently large separation between 0 and $\min\{\tau_e,\tau_m\}$

Excited States



- $\tau_m - \tau_e$ fixed at 0.06 fm
- Excited state contamination becomes $\sim 1\%$ by $\tau_e = 0.7$ fm

Ensembles Used



Ensembles Used

$L^3 \times T$	a (fm)	N_{cfg}	N_{src}	N_{Ψ}	N_{prop}
$24^3 \times 48$	0.0813	650	12	2	312,000
$32^3 \times 64$	0.0600	450	10	3	270,000
$40^3 \times 80$	0.0502	250	6	4	120,000
$48^3 \times 96$	0.0407	341	10	5	341,000

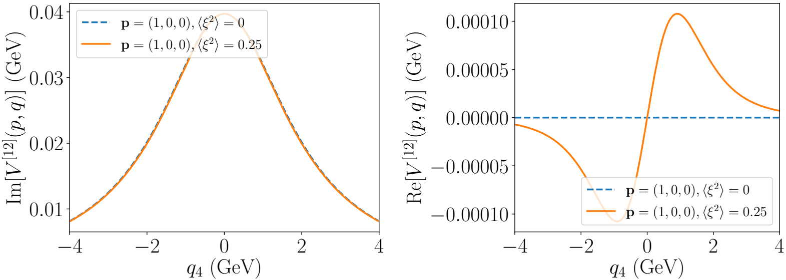
- Quenched approximation with $m_\pi = 550$ MeV
  - Fine dynamical ensembles prohibitively expensive
  - Total compute time: $O(10^5)$ KNL node-hours
- Wilson-clover fermions with non-perturbatively tuned c_{SW}
- With clover term, results fully $O(a)$ improved
  - Axial current renormalizes multiplicatively: $A^\mu \rightarrow A^\mu Z_A(1 + \tilde{b}_A a \tilde{m}_q)$
  - This only affects overall normalization (not $\langle \xi^2 \rangle$)

Choice of Kinematics

$$V^{\mu\nu}(p,q) = \frac{2i f_\pi \varepsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma}{\tilde{Q}^2} \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{\tilde{\omega}^n}{2^n(n+1)} C_W^{(n)}(\tilde{Q}, m_\Psi, \mu)\langle \xi^n \rangle(\mu) + O\left(\frac{\Lambda_{\text{QCD}}}{\tilde{Q}}\right)$$

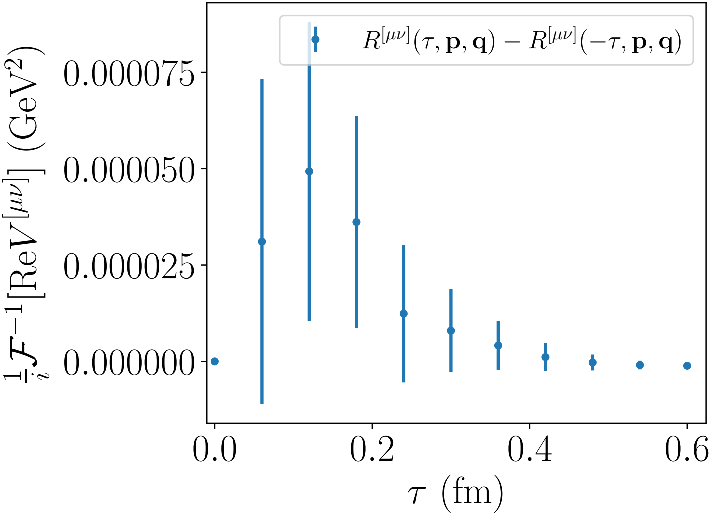
- Wilson coefficients $C_W^{(n)}(\mu = 2 \text{ GeV})$ calculated to 1-loop (hep-lat/2103.09529)
- Fit parameters: f_π, m_Ψ, $\langle \xi^2 \rangle$
- Contribution of second moment $\langle \xi^2 \rangle$ suppressed by $\tilde{\omega}^2$

Choice of Kinematics



$$\mathbf{p} = (1,0,0) = (0.64\ \mathrm{GeV}, 0, 0)$$
$$2\mathbf{q} = (1,0,2) = (0.64\ \mathrm{GeV}, 0, 1.28\ \mathrm{GeV})$$

Noise Reduction



Noise Reduction

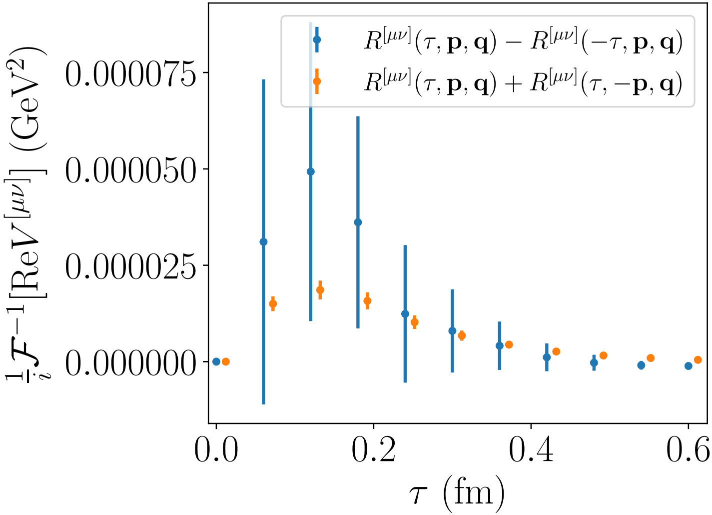
By γ_5-hermiticity of quark propagators,

$$C_3^{\mu\nu}(\tau_e, \tau_m; \mathbf{p}_e, \mathbf{p}_m)^* = C_3^{\nu\mu}(\tau_m, \tau_e; \mathbf{p}_m, \mathbf{p}_e)$$

As a result,

$$\begin{aligned}\mathrm{Re}[V^{\mu\nu}(\mathbf{p}, q)] &= \int_0^\infty d\tau \, [R^{\mu\nu}(\tau; \mathbf{p}, \mathbf{q}) - R^{\mu\nu}(-\tau; \mathbf{p}, \mathbf{q})] \sin(q_4\tau) \\ &= \int_0^\infty d\tau \, [R^{\mu\nu}(\tau; \mathbf{p}, \mathbf{q}) + R^{\mu\nu}(\tau; -\mathbf{p}, \mathbf{q})] \sin(q_4\tau)\end{aligned}$$

Noise Reduction



Fitting Procedures

Two decisions must be made for fitting procedure:

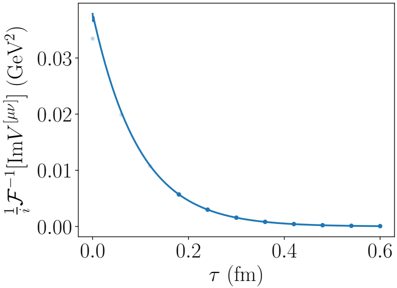
1. Momentum space versus time-momentum representation
2. Order of continuum extrapolation and fit

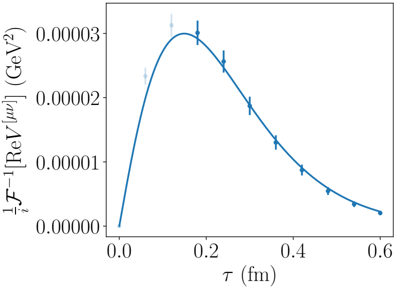
Primary extraction in time-momentum representation with continuum extrapolation at end

- Opposite set of choices (momentum space, continuum extrapolation first) done as cross-check

Fitting Hadronic Tensor

- Fit ratio of 2- and 3-point correlators to inverse FT of OPE



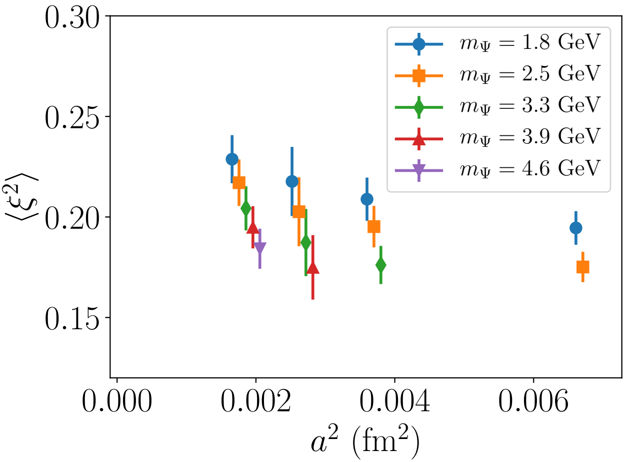


$f_\pi = 149 \pm 1$ MeV

$m_\Psi = 1.85 \pm 0.01$ GeV

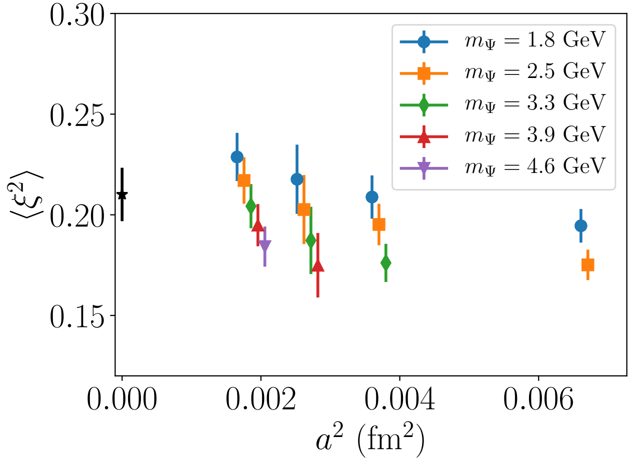
$\langle \xi^2 \rangle = 0.209 \pm 0.011$

Fits to Various Ensembles



Masses are (left to right) $\{1.8, 2.5, 3.3, 3.9, 4.6\}$ GeV

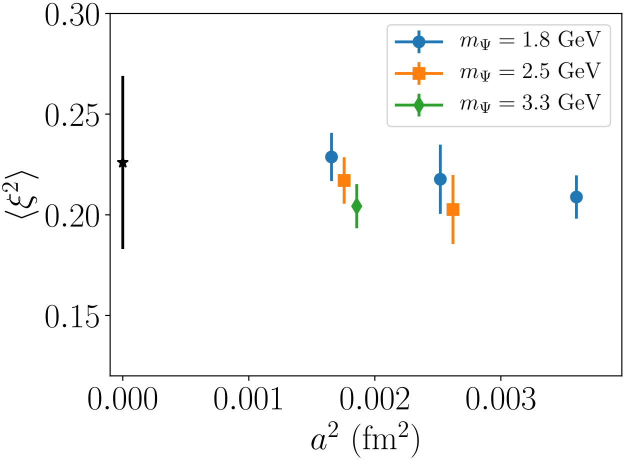
Continuum Extrapolation



$$\text{data} = \langle \xi^2 \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2$$

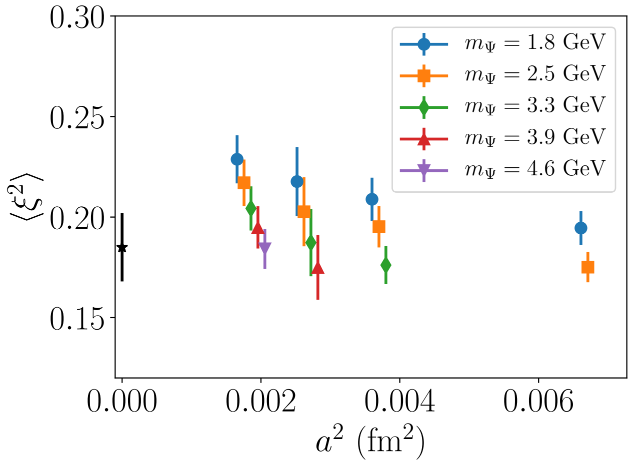
Extrapolate away both discretization errors and twist-3 effects

Uncertainty in Continuum Extrapolation



- Original fit restricted am_Ψ to < 1.05
- Could take a more conservative threshold, e.g. $am_\Psi < 0.7$

Uncertainty in Higher-Twist Effects



- Could add twist-4 term to fit as well

$$\text{data} = \langle \xi^2 \rangle + A m_\Psi^{-1} + B m_\Psi^{-2} + C a^2 + D a^2 m_\Psi + E a^2 m_\Psi^2$$

Remaining Uncertainties

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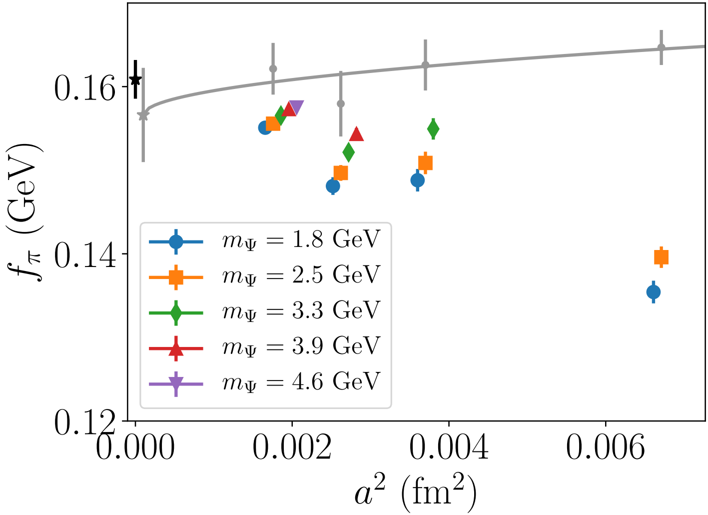
- Quenching: Formally uncontrolled, typically around **10–20%**

Combined Uncertainty

$$\begin{aligned}
\langle \xi^2 \rangle = 0.210 &\pm 0.013 \text{ (statistical)} \\
&\pm \mathbf{0.016} \text{ (continuum)} \\
&\pm \mathbf{0.025} \text{ (higher twist)} \\
&\pm 0.002 \text{ (excited states)} \\
&\pm 0.0002 \text{ (finite volume)} \\
&\pm 0.014 \text{ (unphysically heavy pion)} \\
&\pm 0.002 \text{ (fit range)} \\
&\pm 0.008 \text{ (running coupling)} \\
\hline
\langle \xi^2 \rangle = 0.210 &\pm 0.036 \text{ (total, exc. quenching)}
\end{aligned}$$

f_π Determination



Momentum-Space Method

- Perform discrete Fourier transform of lattice data

$$V^{\mu\nu}(p,q;a) = a \sum_{\tau=-\tau_{\max}}^{\tau_{\max}} e^{i\tau q_4} R^{\mu\nu}(\tau,\mathbf{p},\mathbf{q};a)$$

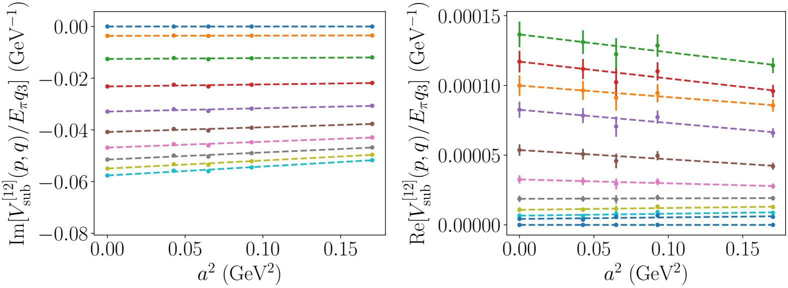
- Consider differences to remove effect of small-τ data

$$V^{\mu\nu}_{\text{sub}}(p,q;a) = V^{\mu\nu}(p,q;a) - V^{\mu\nu}(p,(\mathbf{q},q_{4,\text{sub}});a)$$

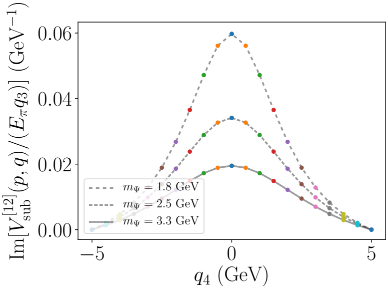
- Extrapolate $V^{\mu\nu}_{\text{sub}}$ to continuum at various q_4 values
- Fit to OPE + model of higher-twist terms and extract $\langle \xi^2 \rangle$

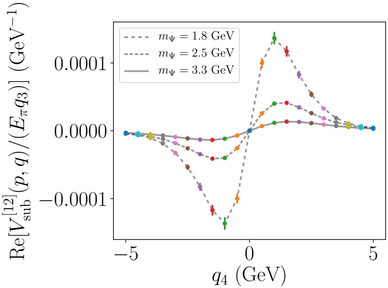
Continuum Extrapolation ($m_\psi = 1.8$ GeV)

$$V^{\mu\nu}_{\text{sub}}(p,q;a) = V^{\mu\nu}_{\text{sub}}(p,q) + a^2 V^{\mu\nu(2)}_{\text{sub}}(p,q) + O(a^3)$$



Results of Continuum Extrapolation





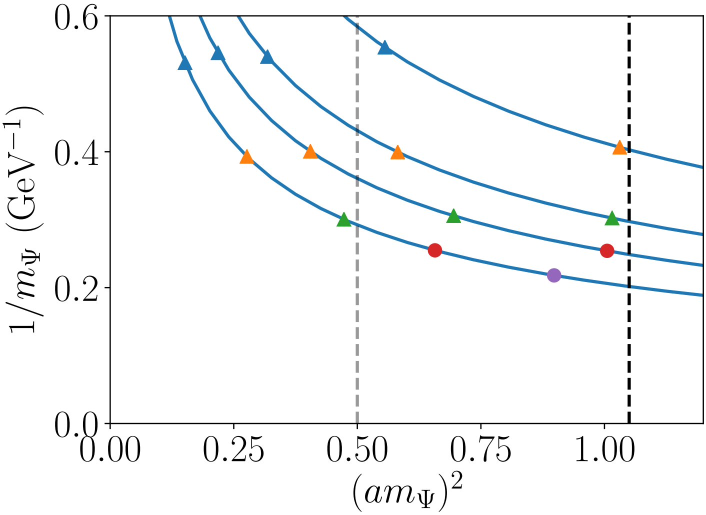
Extraction of $\langle \xi^2 \rangle$

- Add model of higher-twist term to OPE

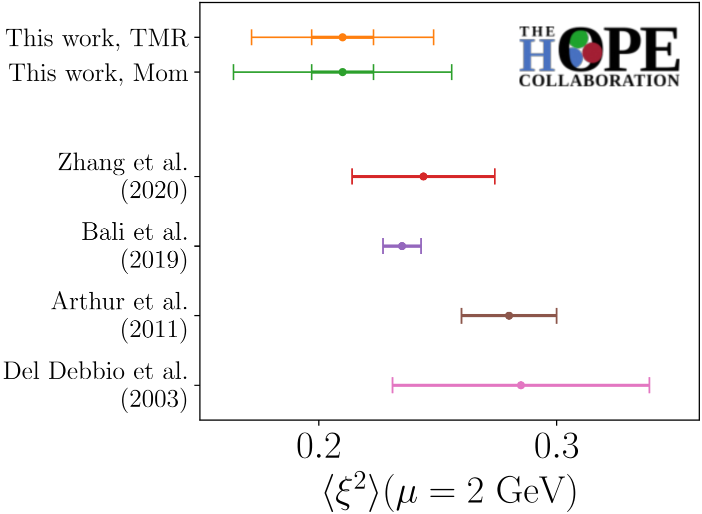
$$\frac{V^{[12]}(q,p,m_\Psi)}{(E_\pi q_3)} = -\frac{2if_\pi}{\tilde{Q}^2}\left\{C_W^{(0)}(\tilde{Q}^2,\mu,\tau) + C_W^{(2)}(\tilde{Q}^2,\mu,\tau)\langle \xi^2 \rangle \left[\frac{\zeta^2 C_2^2(\eta)}{12\tilde{Q}^2}\right]\right\} + \frac{2if_\pi \Lambda_{\text{QCD}}}{\tilde{Q}^3}\left\{b_0 + b_2\left[\frac{\zeta^2 C_2^2(\eta)}{12\tilde{Q}^2}\right]\right\}.$$

- Perform momentum subtraction to OPE to match data
- Fit $f_\pi, \{m_\Psi\}, \langle \xi^2 \rangle, b_0, b_2$
- Result: $\langle \xi^2 \rangle = 0.210 \pm 0.046$ (stat. + sys.)
- Systematics comparable to time-momentum analysis
  - Higher-twist effects less controlled (only three m_Ψ usable)

Ensembles Used



Comparison to Literature



$\langle \xi^4 \rangle$ Calculation

Motivation for $\langle \xi^4 \rangle$

- After $\langle \xi^2 \rangle$, $\langle \xi^4 \rangle$ most phenomenologically important:

$$\phi_\pi(\xi, \mu^2) = \frac{3}{4}(1 - \xi^2) \sum_{n=0}^{\infty} \phi_n(\mu^2) \mathcal{C}_n^{(3/2)}(\xi)$$

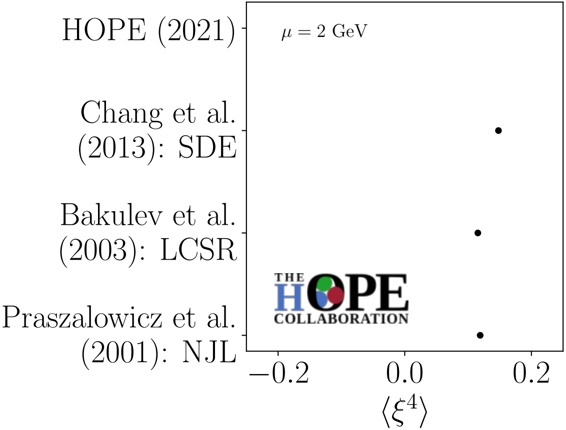
- Scale dependence:

$$\phi_n(\mu_2) = \phi_n(\mu_1) \left(\frac{\alpha_S(\mu_2)}{\alpha_S(\mu_1)} \right)^{\gamma_n/\beta_0} \implies \phi_n(\mu \to \infty) \to 0, \ (n \neq 0)$$

- $\phi_2 = \frac{7}{12}(5 \langle \xi^2 \rangle - \langle \xi^0 \rangle), \phi_4 = \frac{11}{24}(21 \langle \xi^4 \rangle - 14 \langle \xi^2 \rangle + \langle \xi^0 \rangle)$
- Local matrix element: power divergences beyond $\langle \xi^2 \rangle$
- HOPE method: allows arbitrary number of moments (restricted by numerical precision)

State of the field

- Range of theoretical calculations: no lattice calculations of moment.



High Momentum $\implies$ More Moments

- All methods require high momentum to access partonic physics.

- OPE proportional to

$$V^{\mu\nu}(p,q) \sim \sum_{n=0}^{\infty} \langle \xi^n \rangle \tilde{\omega}^n$$

$$\tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2} = \frac{1}{\tilde{x}}$$

- Evaluate HOPE for $|\tilde{\omega}| < 1$.
- Utilize momentum smearing

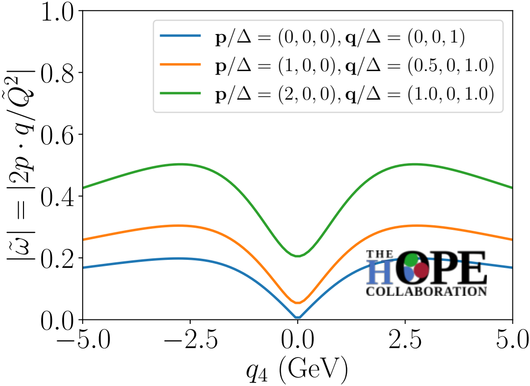


Figure: $\Delta = 0.64$ GeV

Lattice Details

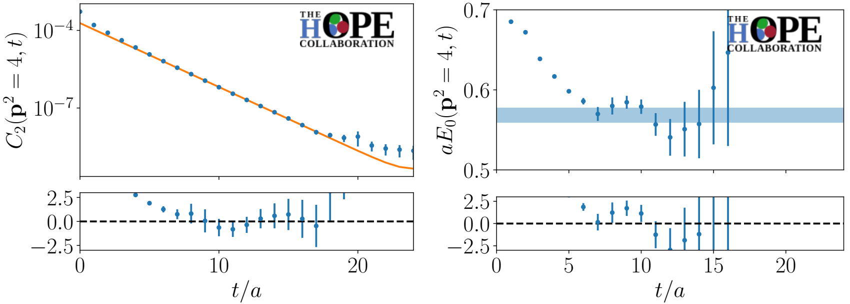
- All calculations performed in CHROMA
- exploratory study: one lattice volume
  - $L = 24$
  - $a = 0.08$ fm
  - Heavy Quark masses (renormalized): $m_\Psi \approx 1.8, 3.2$ GeV
- Configurations quenched
- Wilson clover fermions: Order-a improved
- $m_\pi \sim 550$ MeV, $L_{\text{phys}} \sim 1.92$ fm, $m_\pi L \sim 5.4$

a (fm)	$L^3 \times T$	m_Ψ (GeV)	N_{cfg}	Light Props	Heavy Props
0.08	24×48	1.8	3150	12,600	100,800
0.08	24×48	3.2	3150		100,800

Two-Point Analysis

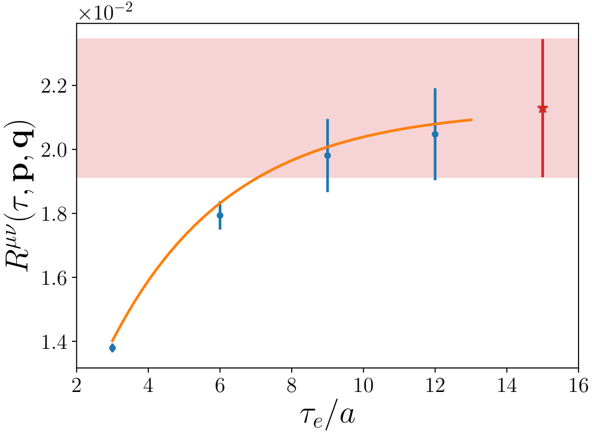
$$R^{\mu\nu}(\tau_e, \tau_m, \mathbf{p}_e, \mathbf{p}_m) = |\langle\Omega|\, \mathcal{O}_\pi^\dagger\, |\pi(\mathbf{p}_e + \mathbf{p}_m)\rangle|\, \frac{C_3^{\mu\nu}(\tau_e, \mathbf{p}_e, \tau_m, \mathbf{p}_m)}{C_2((\tau_e + \tau_m)/2, \mathbf{p}_e + \mathbf{p}_m)}$$

- $n_{\text{cfg}} = 3150$
- $|\mathbf{p}| = 2 \times 0.64$ GeV ~ 1.3 GeV



Excited state contamination $\mathbf{p} \sim 1.3$ GeV

- $R^{\mu\nu}(\tau, \tau_e, \mathbf{p}, \mathbf{q}) = R^{\mu\nu}(\tau, \mathbf{p}, \mathbf{q})\left[1 + Be^{-(E_\pi - E)\tau_e}\right] \underset{\text{large } \tau_e}{\longrightarrow} R^{\mu\nu}(\tau, \mathbf{p}, \mathbf{q})$
- L=24, High momentum ($|\mathbf{p}| = 2$)



Heavy quark Operator Product Expansion

- Perform operator product expansion:

$$\tilde{Q}^2 = q^2 + m_\Psi^2 \qquad \text{large scale}$$

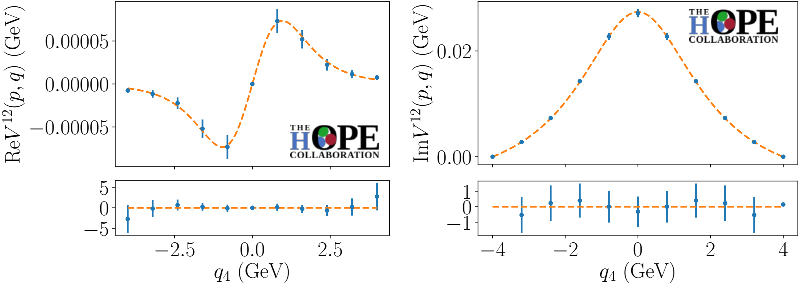
$$\tilde{\omega} = \frac{1}{\tilde{x}} = \frac{2p\cdot q}{\tilde{Q}^2} \qquad \text{expansion parameter}$$

$$V_\pi^{\mu\nu}(p,q) = \frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta}{\tilde{Q}^2} \sum_{\text{even}} C_W^{(n)}\left(\tilde{Q}^2, \mu, m_\Psi\right) \langle \xi^n \rangle \left[\frac{\zeta^n C_n^2(\eta)}{2^n(n+1)}\right] + \underbrace{\mathcal{O}(1/\tilde{Q}^3)}_{\text{Higher twist}}$$

- We fit f_π, m_Ψ^2, $\langle \xi^2 \rangle$, $\langle \xi^4 \rangle$
- Moments $\langle \xi^n \rangle$ unchanged.
- where $\zeta = \sqrt{p^2 q^2}/\tilde{Q}^2$, $\eta = p\cdot q/\sqrt{p^2 q^2}$
- $C_W^{(n)}(\tilde{Q}^2, \mu, \tau)$ determined in arXiv:2103.09529

Low momentum data $L = 24$, $a = 0.08$ fm

$$V_\pi^{\mu\nu}(p,q) = \frac{2i f_\pi \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta}{\tilde{Q}^2} \sum_{\text{even}} C_W^{(n)}\left(\tilde{Q}^2, \mu, m_\Psi\right) \langle \xi^n \rangle \left[\frac{\zeta^n C_n^2(\eta)}{2^n(n+1)}\right]$$



- $\mathbf{p} = (1,0,0)$, $\mathbf{q} = (-1/2, 0, -1)$
- Extract parameters: $f_\pi = 0.129 \pm 0.003$, $m_\Psi = 1.75 \pm 0.02$ GeV, $\langle \xi^2 \rangle = 0.21 \pm 0.03$

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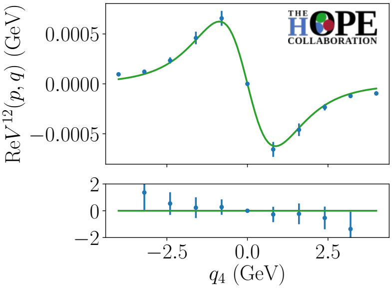
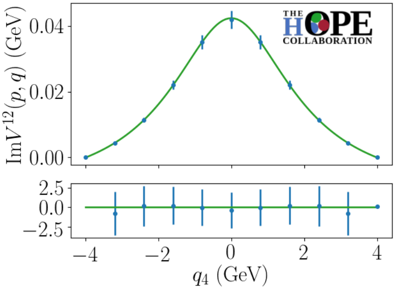
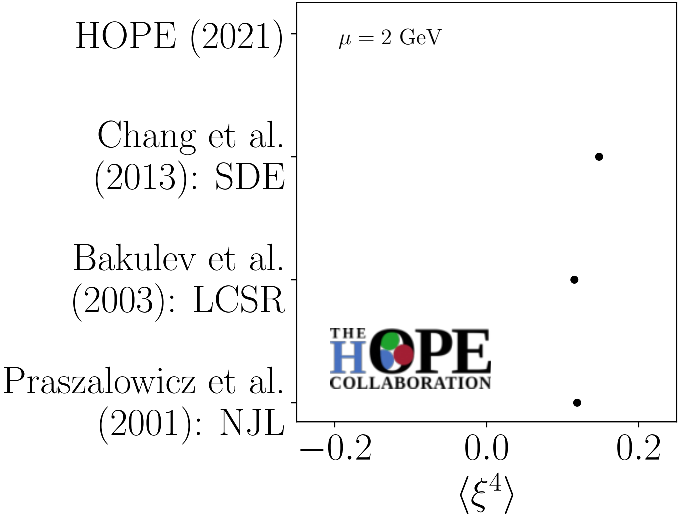




- $\mathbf{p} = (2,0,0),\ \mathbf{q} = (-1,0,-1)$
- Extract parameter: $\langle \xi^4 \rangle = 0.18 \pm 0.06$

Preliminary Comparison



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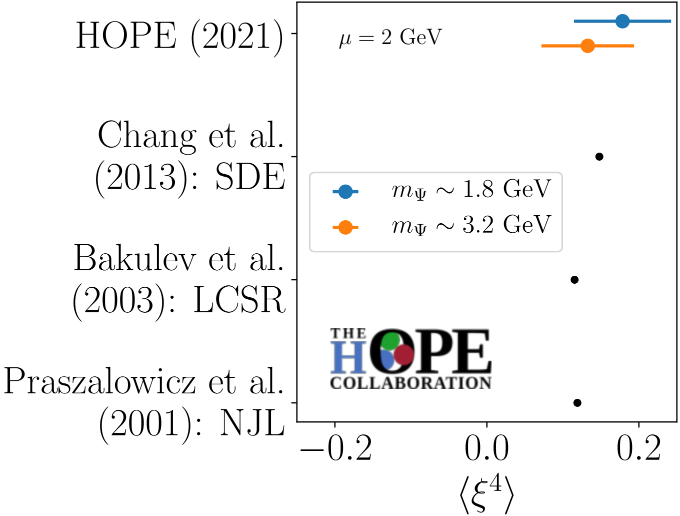


Further Plans and Conclusions

- Analysis is ongoing: results preliminary.
- Heavy quark mass provides additional lever to study higher-twist.

Going forward...

- Required: Excited state contamination, continuum extrapolation, higher-twist,...
- Like to study $|\mathbf{p}| = 3$: further kinematic enhancement
  - Use all data to constrain $\langle \xi^4 \rangle$

Backup Slides

Hadronic Matrix Elements

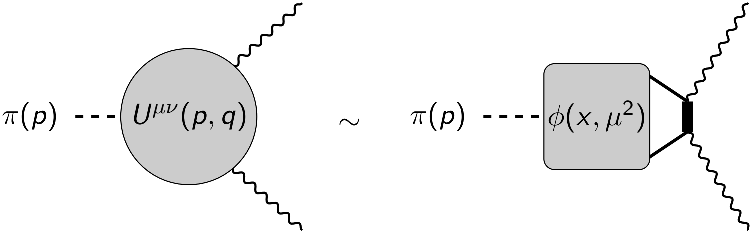
- Study $U^{\mu\nu} = (T^{\mu\nu} - T^{\nu\mu})/2$ where

$$T^{\mu\nu}(p,q) = \int d^4 z e^{iq\cdot z} \langle\Omega|\, T\{J^\mu(z/2)J^\nu(-z/2)\}\, |\pi(\mathbf{p})\rangle$$

- Replace current J^μ with heavy-light current:

$$J^\mu_\Psi(x) = \overline{\Psi}(x)\Gamma^\mu\psi(x) + \overline{\psi}(x)\Gamma^\mu\Psi(x), \qquad \Gamma^\mu = \gamma^\mu\ ,\ \gamma^\mu\gamma_5$$

- $\Psi(x)$ is a fictitious quenched heavy quark species.



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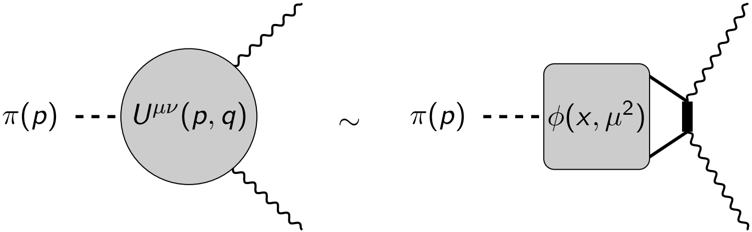

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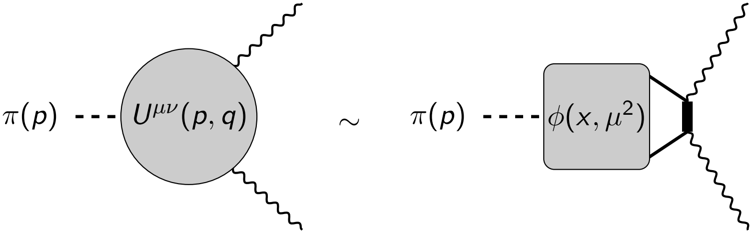

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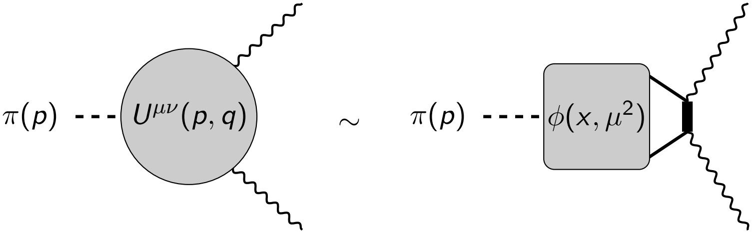
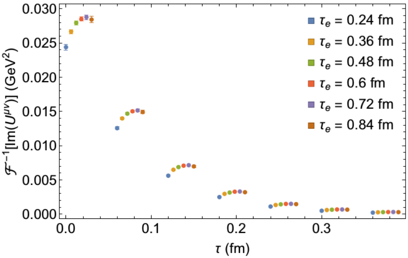


Excited state contamination: $\mathbf{p} = 3$

- $R^{\mu\nu}(\tau, \tau_e, \mathbf{p}, \mathbf{q}) = R^{\mu\nu}(\tau, \mathbf{p}, \mathbf{q}) \left[1 + B e^{-(E_\pi - E)\tau_e}\right] \underset{\text{large } \tau_e}{\longrightarrow} R^{\mu\nu}(\tau, \mathbf{p}, \mathbf{q})$

- L=32, Low momentum ($|\mathbf{p}| = 1$)



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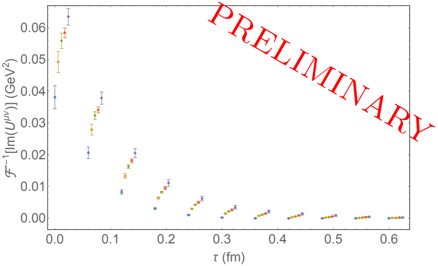

3-Point Function Calculation

- Utilize a sequential source: Fix momentum insertion at $\mathbf{p}_e$