



A view on the proton mass: Sigma terms, the trace anomaly and all that

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Trace anomaly and the proton mass

TRACE ANOMALY

- Classical masses QCD is invariant under scale transformations

$$x \rightarrow \lambda x, q(x) \rightarrow \lambda^{3/2} q(\lambda x), A_\mu(x) \rightarrow \lambda A_\mu(\lambda x) \quad (\text{dilatations})$$

- Quantization/renormalization generates a scale Λ_{QCD} that breaks scale invariance: **dimensional transmutation**
- ⇒ **trace anomaly**

$$\theta_\mu^\mu = \frac{\beta_{\text{QCD}}}{2g} G_{\mu\nu}^a G_a^{\mu\nu} + m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s + \dots$$

- * trace anomaly = signal for the *generation of hadron masses*
- * the mass of any hadron made of light quarks mass is essentially **field energy** (“binding”)

“Mass without mass” (Wheeler, 1962)

anomalous dimension γ_m neglected for simplicity

ANATOMY of the NUCLEON MASS

$$\begin{aligned}
 m_N &= \langle N(p) | \theta_\mu^\mu | N(p) \rangle \\
 &= \langle N(p) | \underbrace{\frac{\beta_{\text{QCD}}}{2g} G_{\mu\nu}^a G_a^{\mu\nu}}_{\text{field energy}} + \underbrace{m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s}_{\text{Higgs}} | N(p) \rangle
 \end{aligned}$$

- Dissect the various contributions:

★ $\langle N(p) | m_u \bar{u}u + m_d \bar{d}d | N(p) \rangle = 40 \dots 70 \text{ MeV} \doteq \sigma_{\pi N}$

from the analysis of the pion-nucleon sigma term & lattice QCD (before 2015)

Gasser, Leutwyler, Sainio; Borasoy & M., Büttiker & M., Pavan et al., Alarcon et al. . . .

★ $\langle N(p) | m_s \bar{s}s | N(p) \rangle = 20 \dots 60 \text{ MeV}$ from lattice → summary

⇒ bulk of the nucleon mass is generated by the gluon fields / field energy

⇒ this is a central result of QCD

⇒ requires better Roy-Steiner analysis of πN and lattice data

→ this talk

ROLE of the PION-NUCLEON σ -TERM

- Scalar couplings of the nucleon:

$$\langle N | m_q \bar{q} q | N \rangle = f_q^N m_N \quad (N = p, n) \\ (q = u, d, s)$$

↪ Dark Matter detection

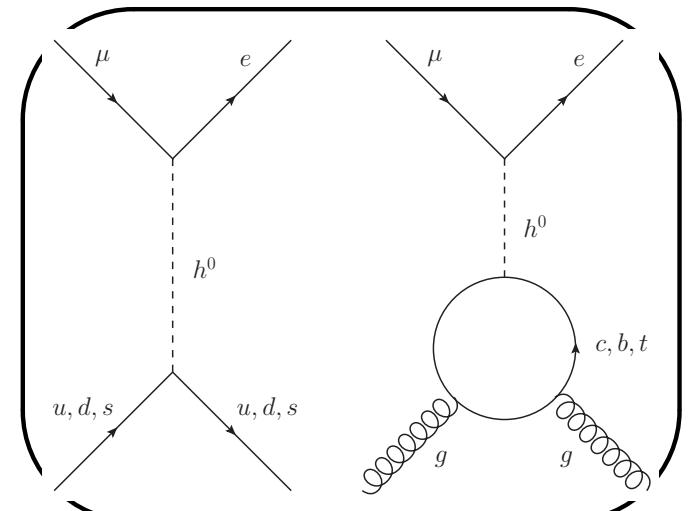
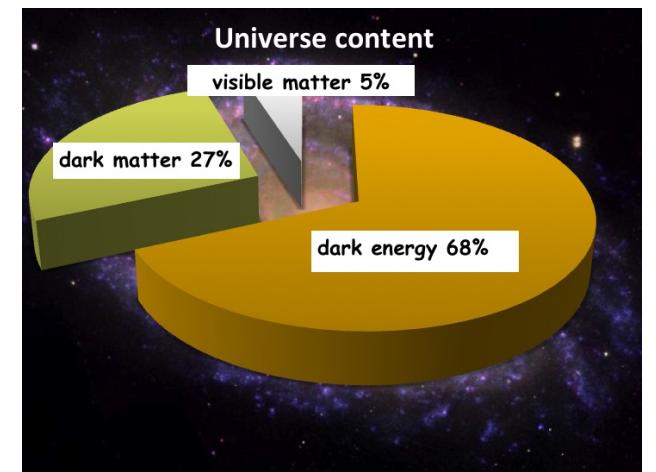
↪ $\mu \rightarrow e$ conversion in nuclei

- Condensates in nuclear matter

$$\frac{\langle \bar{q}q \rangle(\rho)}{\langle 0 | \bar{q}q | 0 \rangle} = 1 - \frac{\rho \sigma_{\pi N}}{F_\pi^2 M_\pi^2} + \dots$$

- CP-violating πN couplings

↪ hadronic EDMs (nucleon, nuclei)



Crivellin, Hoferichter, Procura

σ -term basics

σ -TERM BASICS

- Scalar form factor of the nucleon (isospin limit $\hat{m} = (m_u + m_d)/2$):

$$\sigma_{\pi N}(t) = \langle N(p') | \hat{m}(\bar{u}u + \bar{d}d) | N(p) \rangle , \quad t = (p' - p)^2$$

- Cheng-Dashen Low-Energy Theorem (LET):

Cheng, Dashen (1971)

$$\bar{D}^+(\nu = 0, t = 2M_\pi^2) = \sigma(2M_\pi^2) + \Delta_R \quad \left[\nu = \frac{s-u}{4m_N} \right]$$

- \bar{D}^+ – isospin-even, Born-term subtracted pion-nucleon scattering amplitude

$$\bar{D}^+(0, 2M_\pi^2) = A^+(m_N^2, 2M_\pi^2) - \frac{g_{\pi N}^2}{m_N}$$

→ best determined from πN data using dispersion relations (unphysical region)

- reminder Δ_R , calculated in CHPT to $\mathcal{O}(p^4)$, no chiral logs

$$\Delta_R \lesssim 2 \text{ MeV}$$

Bernard, Kaiser, UGM (1996)

σ -TERM BASICS continued

- Standard decomposition of the σ -term: $\sigma_{\pi N} = \sigma_{\pi N}(0)$

$$\sigma_{\pi N} = \Sigma_d + \Delta_D - \Delta_\sigma - \Delta_R$$

$$\Sigma_d = F_\pi^2 (d_{00}^+ + 2M_\pi^2 d_{01}^+) \quad \rightarrow \text{full RS analysis}$$

$$\left. \begin{aligned} \Delta_D &= \bar{D}^+(0, 2M_\pi^2) - \Sigma_d \\ \Delta_\sigma &= \sigma(2M_\pi^2) - \sigma_{\pi N} \end{aligned} \right\} \rightarrow \text{RS t-channel analysis}$$

- d_{00}^+, d_{01}^+ – subthreshold expansion coefficients (around $\nu = t = 0$)

- Strong $\pi\pi$ rescattering in Δ_D and Δ_σ , the difference is small!

Gasser, Leutwyler, Sainio (1991)

- Most precise analysis of the scalar form factor of the nucleon:

Hoferichter, Ditsche, Kubis, UGM (2012)

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2) \text{ MeV}$$

Hadronic Atoms

WHY HADRONIC ATOMS?

- Hadronic atoms: bound by the static Coulomb force (QED)
- Many species: $\pi^+\pi^-$, $\pi^\pm K^\mp$, $\pi^- p$, $\pi^- d$, $K^- p$, $K^- d$, ...
- Observable effects of QCD: strong interactions as **small** perturbations

★ energy shift ΔE

★ decay width Γ

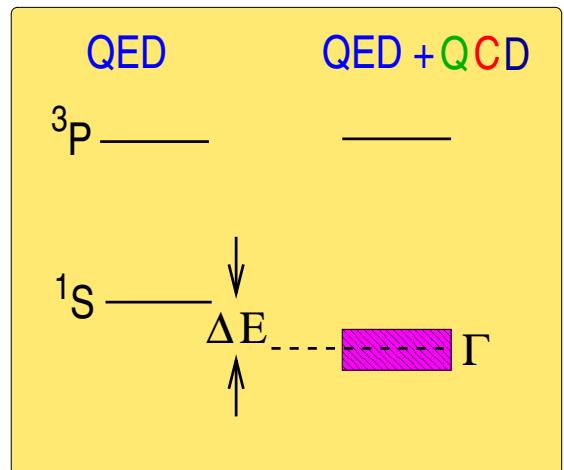
⇒ access to scattering at zero energy!

= S-wave scattering lengths

⇒ best way to determine the scattering lengths!

- can be analyzed in suitable NREFTs

Pionic hydrogen (πH)



Gasser, Rusetsky, ... 2002

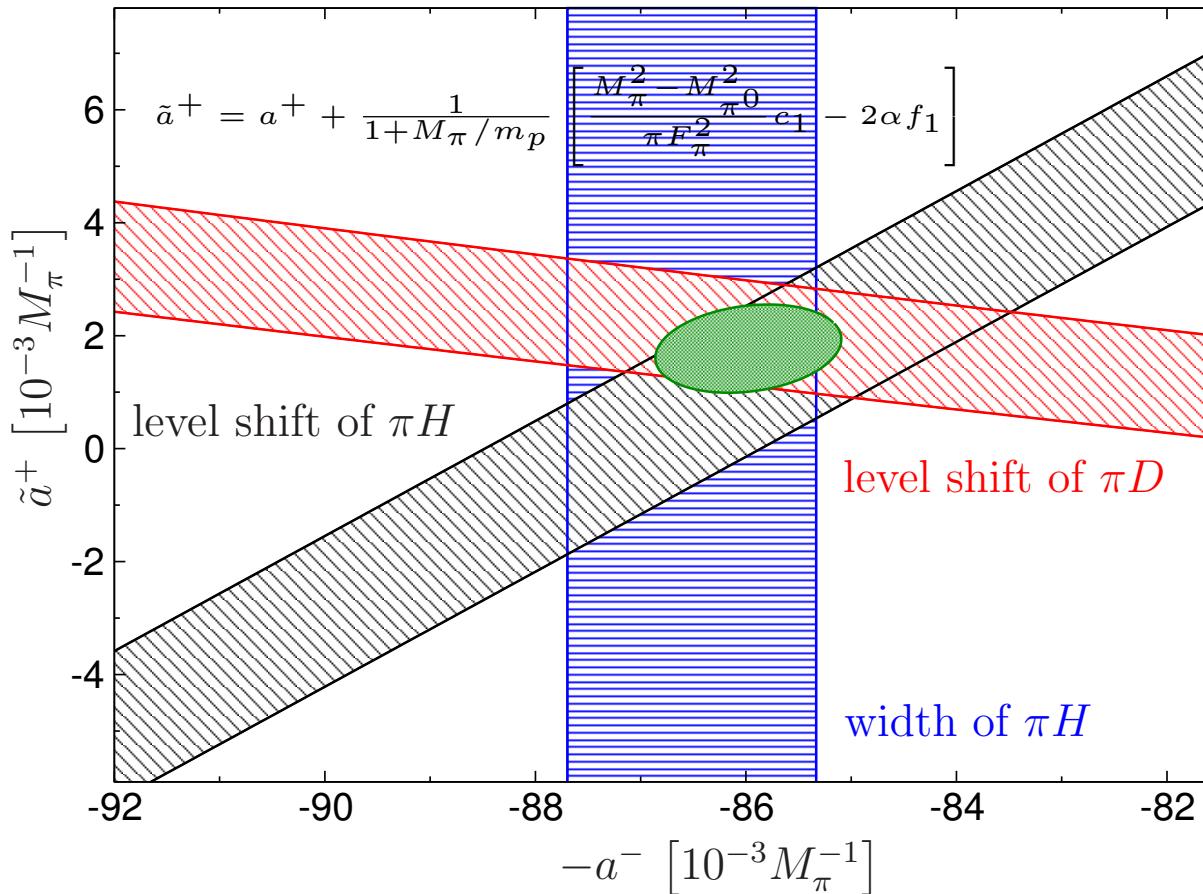
Pionic deuterium (πD)

Baru, Hoferichter, Kubis ... 2011

PION-NUCLEON SCATTERING LENGTHS

- superb experiments performed at PSI

Gotta et al.



- πH level shift $\Rightarrow \pi^- p \rightarrow \pi^- p$
- πD level shift
 \Rightarrow isoscalar $\pi^- N \rightarrow \pi^- N$
- πH width $\Rightarrow \pi^- p \rightarrow \pi^0 n$



$$a^+ = (7.6 \pm 3.1) \cdot 10^{-3}/M_\pi$$

$$a^- = (86.1 \pm 0.9) \cdot 10^{-3}/M_\pi$$

\Rightarrow very precise value for a^- & first time definite sign for a^+

GMO SUM RULE

- Goldberger-Miyazawa-Oehme sum rule:

Goldberger, Miyazawa, Oehme 1955

$$\frac{g_{\pi N}^2}{4\pi} = \left[\left(\frac{m_p + m_n}{M_\pi} \right)^2 - 1 \right] \left\{ \left(1 + \frac{M_\pi}{m_p} \right) \frac{M_\pi}{4} \underbrace{\left(a_{\pi^- p} - a_{\pi^+ p} \right)}_{\text{just determined}} - \frac{M_\pi^2}{2} J^- \right\}$$

$$= 13.69 \pm 0.12 \pm 0.15$$

Baru et al. (2011)

$$J^- = \frac{1}{4\pi^2} \int_0^\infty dk \frac{\sigma_{\pi^- p}^{\text{tot}}(k) - \sigma_{\pi^+ p}^{\text{tot}}(k)}{\sqrt{M_\pi^2 + k^2}}$$

- J^- is very well determined

Ericson et al. 2002, Abaev et al. 2007

- consistent with other determinations:

$$\pi N \quad 13.75 \pm 0.15$$

Arndt et al. 1994

$$NN \quad 13.54 \pm 0.05$$

de Swart et al. 1997

Roy-Steiner equations

Ditsche, Hoferichter, Kubis, UGM, JHEP **1206** (2012) 043

Hoferichter, Ditsche, Kubis, UGM, JHEP **1206** (2012) 063

HYPERBOLIC DISPERSION RELATIONS

- Roy-Steiner equations are hyperbolic dispersion relations (HDRs):

$$(s - a)(u - a) = b, \quad a, b \in \mathbb{R} \quad [b = b(s, t, a)]$$

Steiner (1968), Hite, Steiner (1973)

- why HDRs?

- ↪ combine all *physical regions*
very important for a reliable continuation to the subthreshold region
Stahov (1999)
- ↪ especially powerful for the determination of the σ -term
Koch (1982)
- ↪ $s \leftrightarrow u$ crossing is explicit
- ↪ absorptive parts are only needed in regions where
the corresponding PW expansions converge
- ↪ judicious choice of a allows to increase the range of convergence

RS EQUATIONS: s -CHANNEL

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- s -channel part of the full RS system:

$$f_{l+}^I(W) = N_{l+}^I(W) + n \text{ subtractions around } \nu = t = 0$$

$$\begin{aligned} &+ \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K_{ll'}^I(W, W') \operatorname{Im} f_{l'+}^I(W') + K_{ll'}^I(W, -W') \operatorname{Im} f_{(l'+1)-}^I(W') \right\} \\ &+ \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_J \left\{ G_{lJ}(W, t') \operatorname{Im} f_+^J(t') + H_{lJ}(W, t') \operatorname{Im} f_-^J(t') \right\} \end{aligned}$$

↪ $N_{l+}^I(W)$ nucleon Born term contribution

↪ coupling to s -channel absorptive parts $\sim K_{ll'}^I = \frac{\delta_{ll'}}{W' - W} + \dots$

↪ coupling to t -channel absorptive parts $\sim G_{lJ}, H_{lJ}$

↪ range of convergence: $a = -23.19 M_\pi^2$

$$\Rightarrow s \in [(m_N + M_\pi)^2, 97.30 M_\pi^2] \Leftrightarrow W \in [1.08, 1.38] \text{ GeV}$$

RS EQUATIONS: t -CHANNEL

- t -channel part of the full RS system (only show $f_+^J(t)$):

$$f_+^J(t) = \tilde{N}_+^J(t) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l=0}^{\infty} \left\{ \tilde{G}_{Jl}(t, W') \operatorname{Im} f_{l+}^I(W') + \tilde{G}_{Jl}(t, -W') \operatorname{Im} f_{(l+1)-}^I(W') \right\} \\ + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J'} \left\{ \tilde{K}_{JJ'}^1(t, t') \operatorname{Im} f_+^{J'}(t') + \tilde{K}_{JJ'}^2(t, t') \operatorname{Im} f_-^{J'}(t') \right\} + n \text{ sub. } \forall J \geq 0$$

↪ $\tilde{N}_{l+}^I(W)$ nucleon Born term contribution (no kinematical singularity)

↪ coupling to t -channel absorptive parts $\sim \tilde{K}_{JJ'}^{1,2,3} = \frac{\delta_{JJ'}}{t' - t} + \dots$

↪ only higher t -channel PWs couple to lower ones!

↪ coupling to s -channel absorptive parts $\sim \tilde{G}_{Jl}, \tilde{H}_{Jl}$

↪ range of convergence: $a = -2.71 M_\pi^2$

$$\Rightarrow t \in [4M_\pi^2, 205.45 M_\pi^2] \Leftrightarrow \sqrt{t} \in [0.28, 2.00] \text{ GeV}$$

UNITARITY RELATIONS

- s -channel elastic unitarity ($I_s = 1/2, 3/2$):

$$\text{Im } f_{l\pm}^{I_s}(W) =$$

$$\sqrt{\frac{\lambda(s, m_N^2, M_\pi^2)}{4s}} \left| f_{l\pm}^{I_s}(W) \right|^2 \theta(W - (m_N + M_\pi))$$

- t -channel extended unitarity:
(two-body intermediate states)

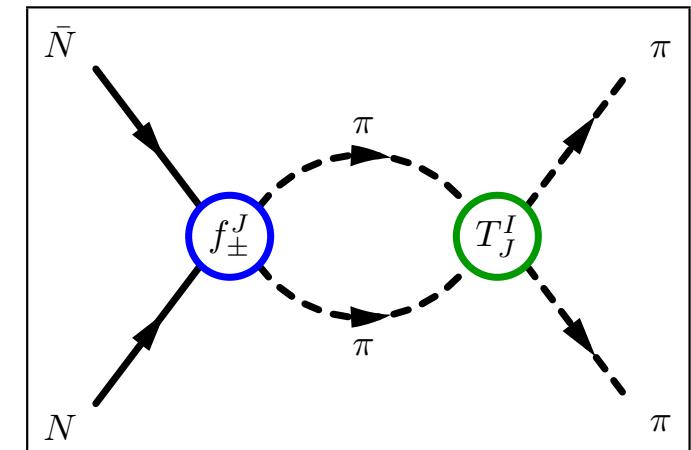
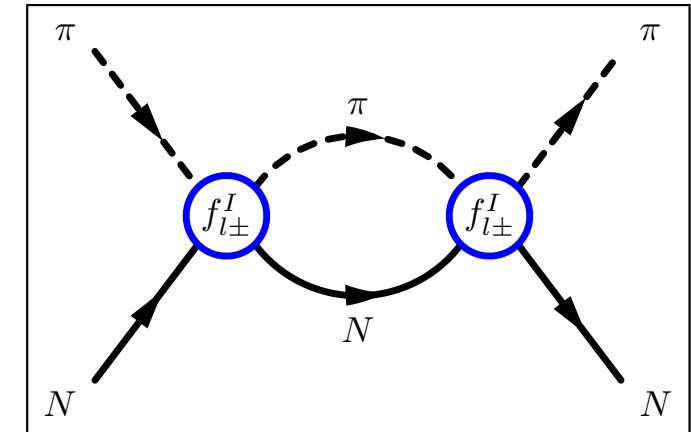
$$\text{Im } f_{l\pm}^J(t) =$$

$$T_J^I(t)^* f_{l\pm}^J \theta(t - 4M_\pi^2)$$

+ $\bar{K}K$ – loop + ...

⇒ Muskhelishvili-Omnès (MO) problem !

⇒ precise $\pi\pi$ input (T_J^I) from Roy equations, Berne and Madrid/Cracow groups



SOLUTION STRATEGY

- RS equations have a limited range of validity:

$$\sqrt{s} \leq \sqrt{s_m} = 1.38 \text{ GeV}$$

$$\sqrt{t} \leq \sqrt{t_m} = 2.00 \text{ GeV}$$

- Input/Constraints

S- and P-waves above the
matching point ($s > s_m, t > t_m$)

Inelasticities

Higher waves (D-, F-, ...)

Scattering lengths from hadronic atoms

- Output

S- and P-wave phase shifts below the
matching point ($s \leq s_m, t \leq t_m$)

Subthreshold parameters

→ Pion-nucleon σ -term

→ LECs of pion-nucleon CHPT

→ N form factor spectral functions

- πN input from SAID/GWU, $\pi\pi$ input from Bern and Madrid/Cracow
- important check: recover KH80 phases with appropriate input

ERROR ANALYSIS

- Variation of the input:

use KH80 input instead of GWU/SAID (higher PWs, inelasticities) → small effect

very small effect from s-channel PWs with $\ell > 5$

small effect from the S-wave extrapolation for $t > 1.3$ GeV

negligible effect of the the ρ' and the ρ''

very significant effect of the D-waves (esp. $f_2(1270)$)

F-waves shown to be negligible

- Other sources of uncertainty:

statistical errors (shallow fit minima)

matching conditions (close to W_m) [no error on SAID, use smoothed KH80]

scattering lengths errors (important for $\sigma_{\pi N}$)

⇒ First time in a dispersive analysis of πN scattering!

Results

Hoferichter, Ruiz de Elvira, Kubis, UGM

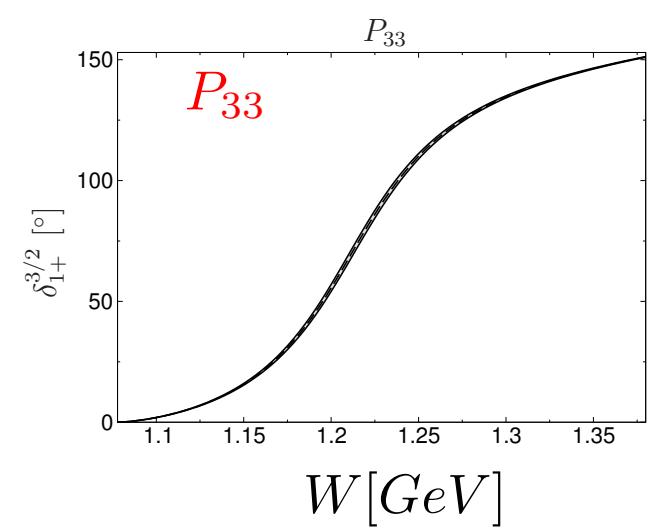
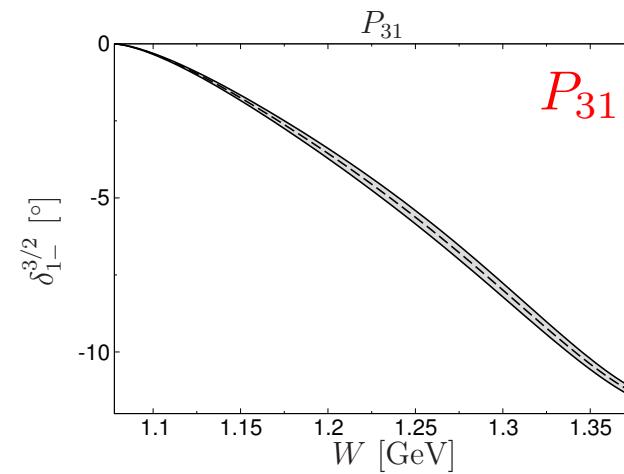
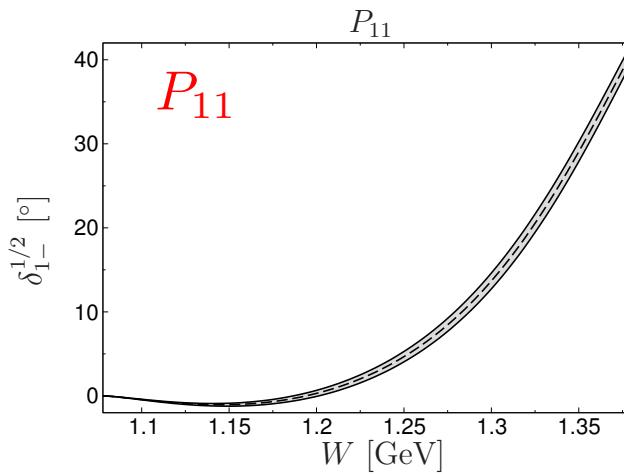
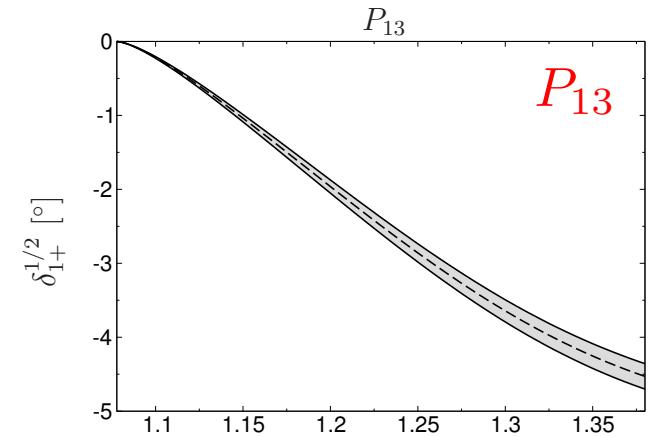
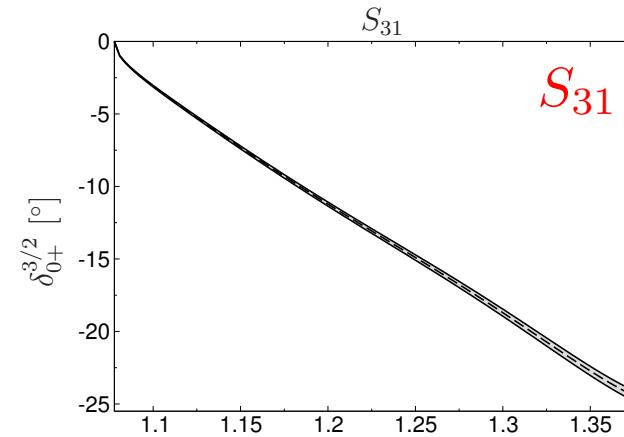
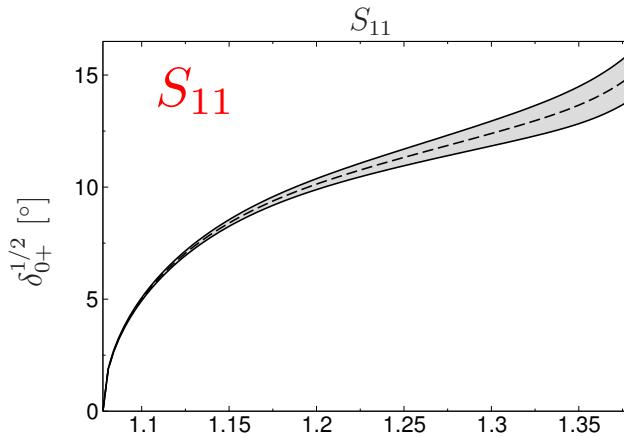
Phys. Rev. Lett. **115** (2015) 092301 [arXiv:1506.04142]

Phys. Rev. Lett. **115** (2015) 192301 [arXiv:1507.07552]

Phys. Rept. **625** (2016) 1 [arXiv:1507.07552]

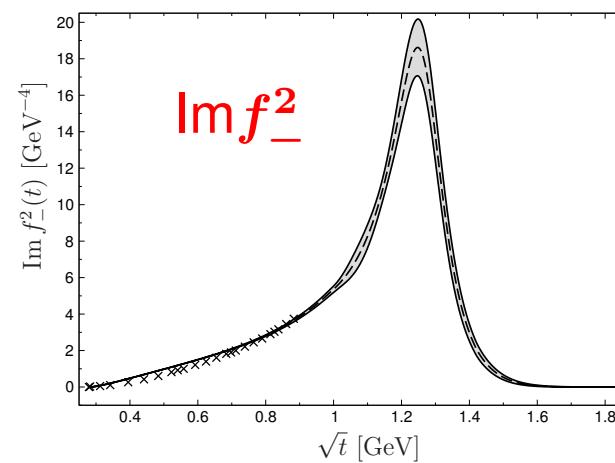
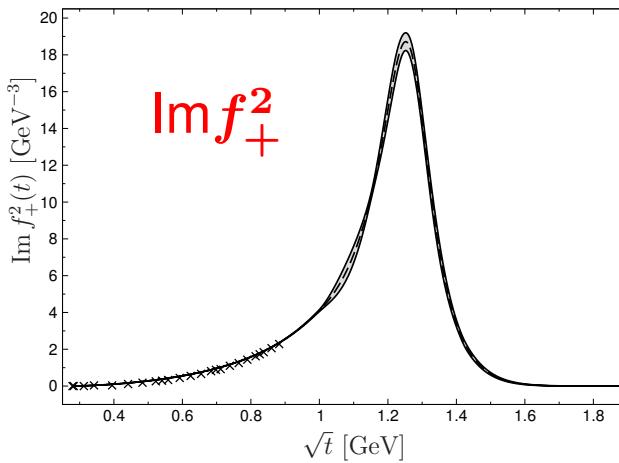
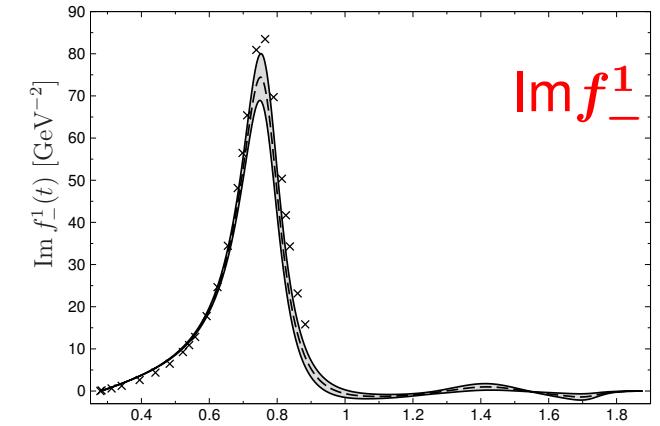
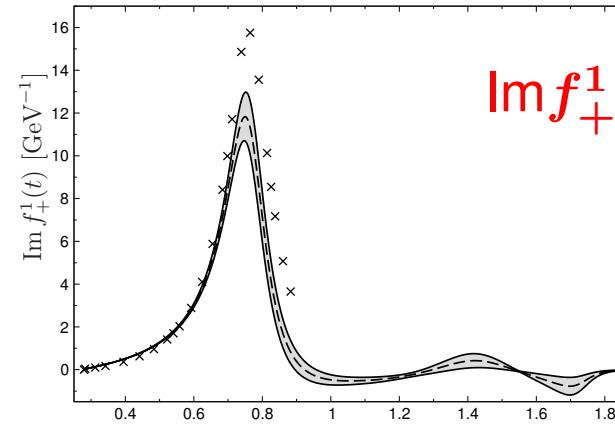
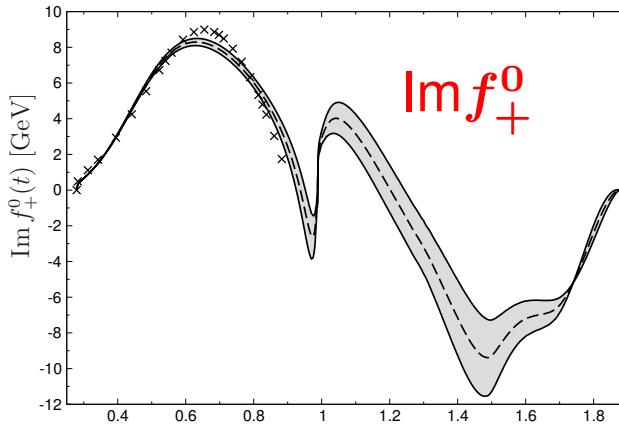
PHASE SHIFTS

- S- and P-waves up to the matching point [Notation: $L_{2I_s 2J}$]



t-CHANNEL PARTIAL WAVES

- Imaginary parts of the t-channel partial waves (cf KH80)



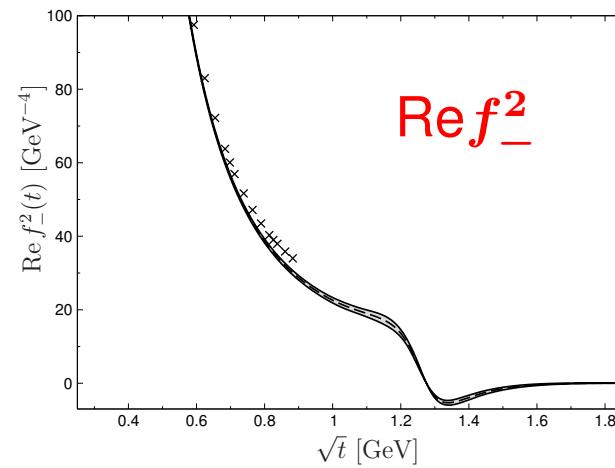
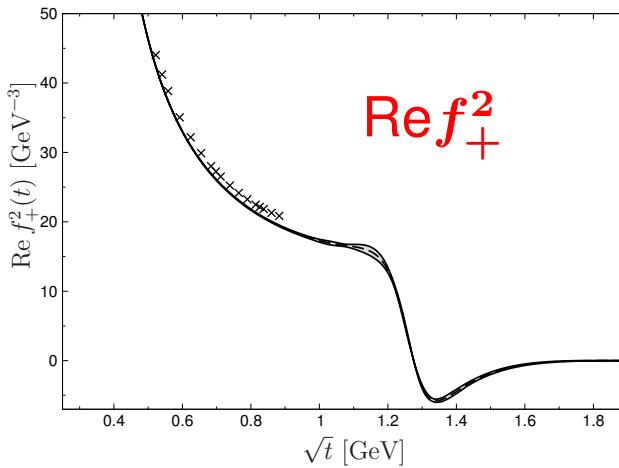
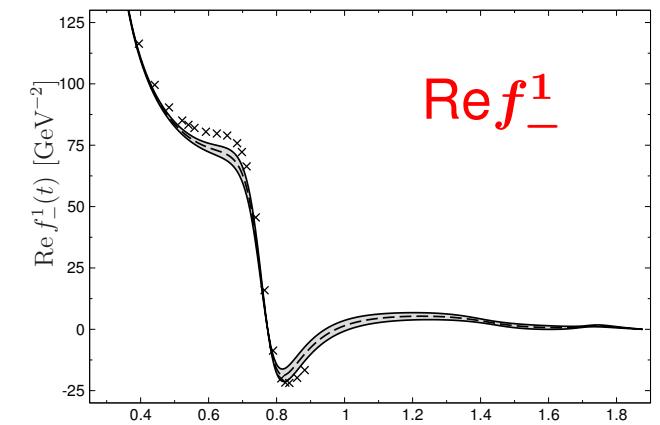
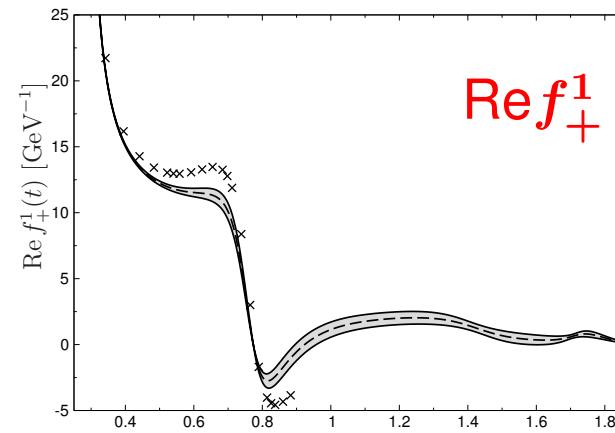
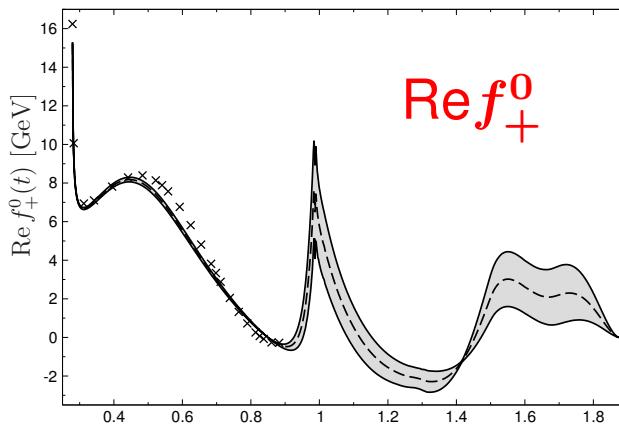
\sqrt{t} [GeV]

$\text{xxx} = \text{KH80}$

reproduced if a^+ , a^-
and $g_{\pi N}$ are readjusted

t-CHANNEL PARTIAL WAVES continued

- Real parts of the t-channel partial waves (cf KH80)



$\text{xxx} = \text{KH80}$

reproduced if a^+ , a^-
and $g_{\pi N}$ are readjusted

RESULTS for the SIGMA-TERM

- Basic formula: $\sigma_{\pi N} = F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D - \Delta_\sigma - \Delta_R$
- Subthreshold parameters output of the RS equations:

$$d_{00}^+ = -1.36(3) M_\pi^{-1} \quad [\text{KH: } -1.46(10) M_\pi^{-1}]$$

$$d_{01}^+ = 1.16(3) M_\pi^{-3} \quad [\text{KH: } 1.14(2) M_\pi^{-3}]$$

- $\Delta_D - \Delta_\sigma = (1.8 \pm 0.2) \text{ MeV}$ Hoferichter, Ditsche, Kubis, UGM (2012)
- $\Delta_R \lesssim 2 \text{ MeV}$ Bernard, Kaiser, UGM (1996)
- Isospin breaking in the CD theorem shifts $\sigma_{\pi N}$ by $+3.0 \text{ MeV}$

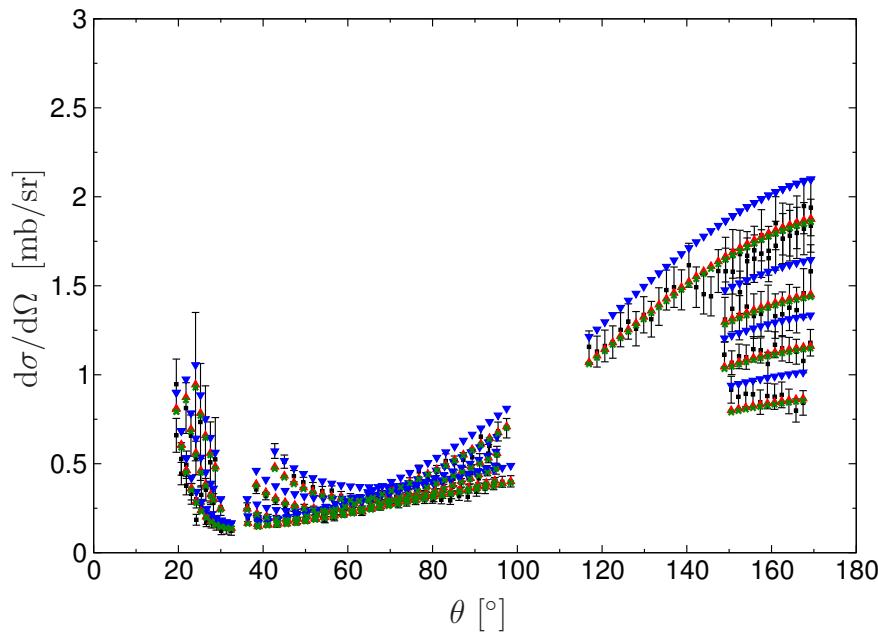
$$\Rightarrow \sigma_{\pi N} = (59.1 \pm 1.9_{\text{RS}} \pm 3.0_{\text{LET}}) \text{ MeV} = (59.1 \pm 3.5) \text{ MeV}$$

[NB: recover $\sigma_{\pi N} = 45 \text{ MeV}$ if KH80 scattering lengths are used]

SANITY CHECK: NO HADRONIC ATOM INPUT

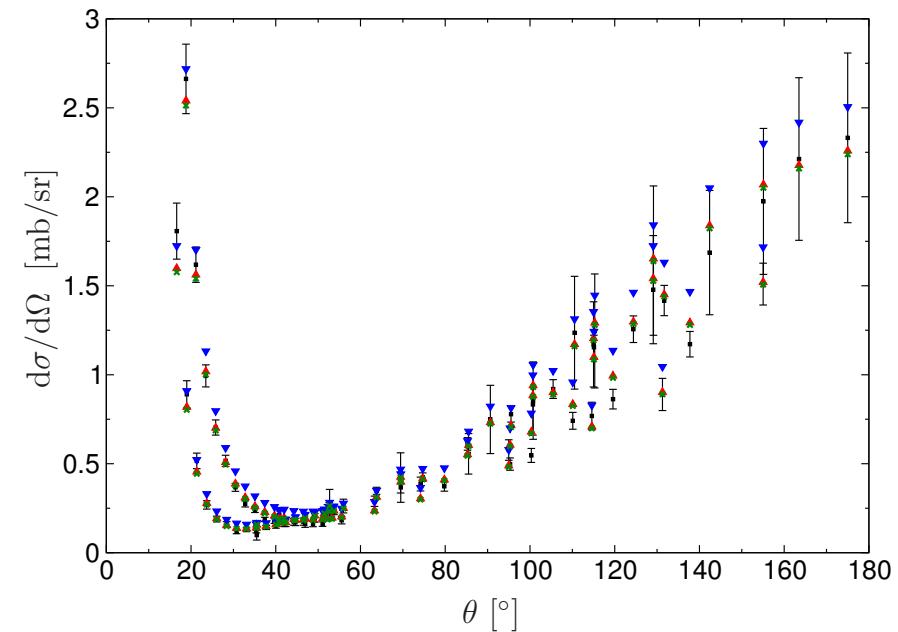
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- Fit to the pion-nucleon data base (GWU), electromagnetic corrections to the data à la Tromberg et al and treat normalizations of the data as fit parameters
- Fit to low-energy data based on the RS representation that are dominated from the scattering lengths (up to $T_{\pi}^{\max} = 33 - 55$ MeV)



black: data with readjusted norm

red triangles: RS solution with hadronic atom input



green triangles: new RS solution

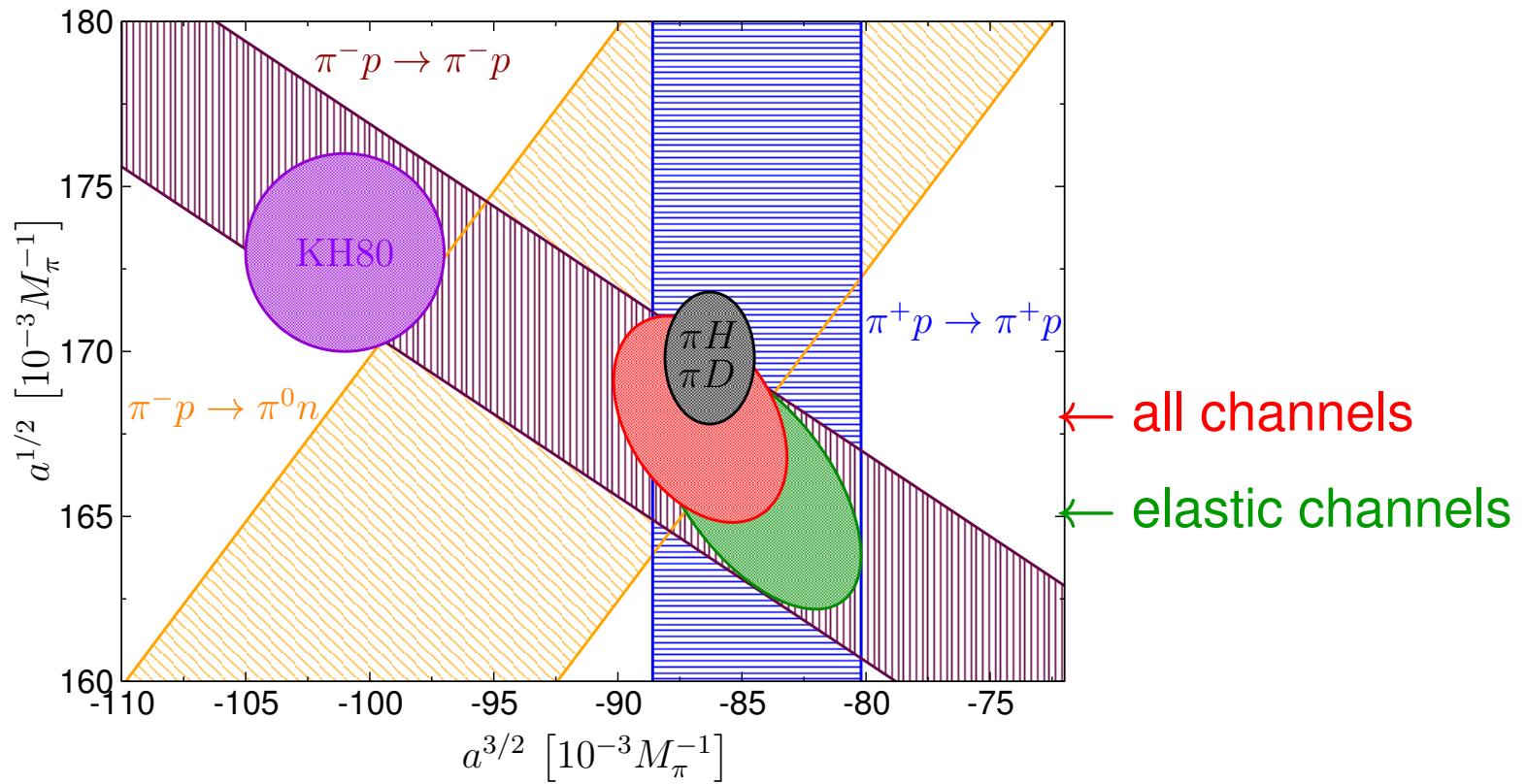
blue triangles: KH80

SANITY CHECK: NO HADRONIC ATOM INPUT cont'd

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- Scattering lengths in $[10^{-3}/M_\pi]$ and σ -term:

$$a^{1/2} = -86.7(3.5), \quad a^{3/2} = 167.9(3.2), \quad \boxed{\sigma_{\pi N} = 58(5) \text{ MeV}}$$



- consistent picture!

[details in Ruiz De Elvira, Hoferichter, Kubis, UGM, J. Phys. G **45** (2018) 024001]

Comparison with recent results from lattice QCD

Hoferichter, Ruiz de Elvira, Kubis, UGM, Phys.Lett. B **760** (2016) 74
[arXiv:1602.07688]

RESULTS for the SIGMA-TERM

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- Recent results from various LQCD collaborations:

collaboration	$\sigma_{\pi N}$ [MeV]	reference
BMWc	38(3)(3)	Dürr et al. (2015)
χ QCD	45.9(7.4)(2.8)	Yang et al. (2015)
ETMC	37.2(2.6)(4.7)	Abdel-Rehim et al. (2016)
RQCD	35(6)	Bali et al. (2016)
ETMC	41.6(3.8)	Alexandrou et al. (2020)
BMWc	37(3)(4)	Borsanyi et al. (2020)

- We seem to have a problem - do we? [we = RS folks]

- Robust prediction of the RS analysis:

$$\sigma_{\pi N} = (59.1 \pm 3.1) \text{ MeV} + \sum_{I_s} c_{I_s} (a^{I_s} - \bar{a}^{I_s}) \quad (I_s = \frac{1}{2}, \frac{3}{2})$$

$$c_{1/2} = 0.242 \text{ MeV} \times 10^3 M_\pi, \quad c_{3/2} = 0.874 \text{ MeV} \times 10^3 M_\pi$$

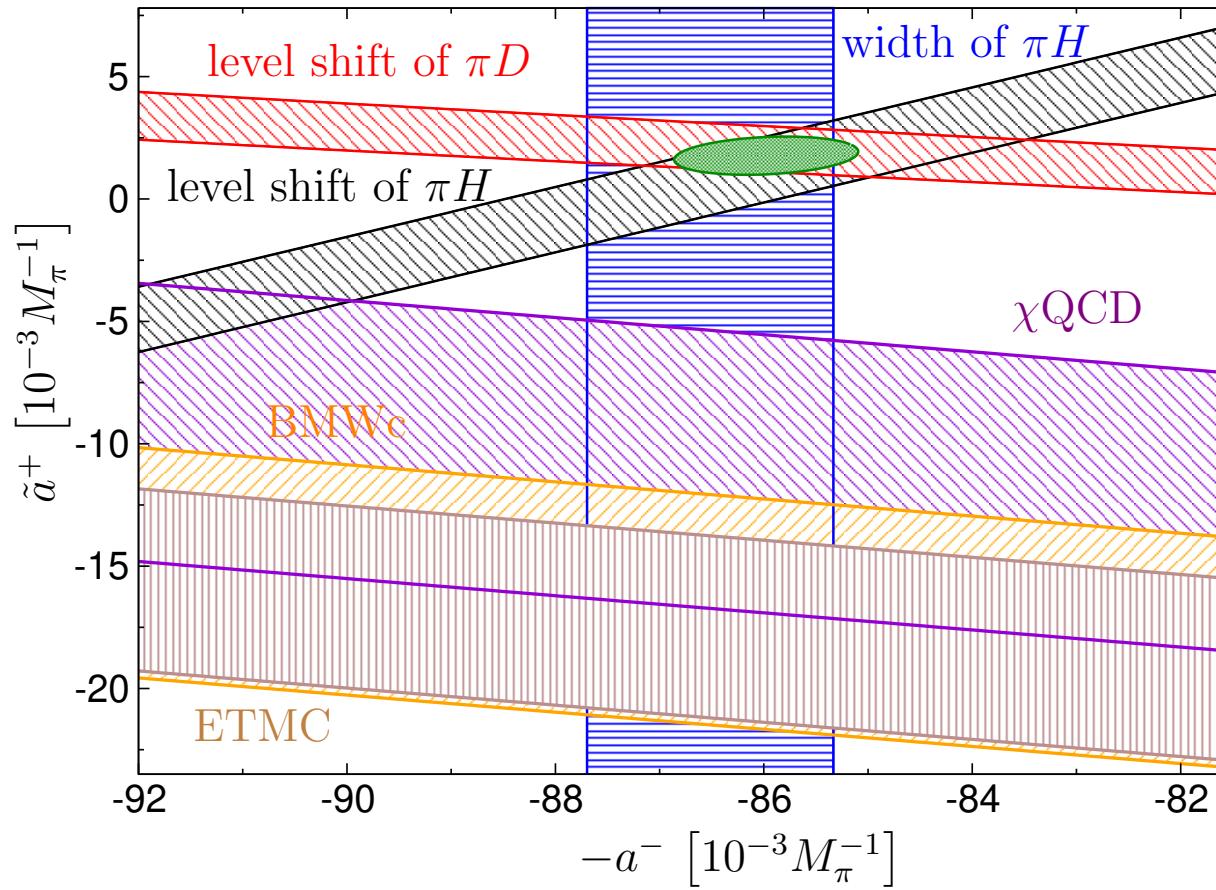
$$\bar{a}^{1/2} = (169.8 \pm 2.0) \times 10^{-3} M_\pi^{-1}, \quad \bar{a}^{3/2} = (-86.3 \pm 1.8) \times 10^{-3} M_\pi^{-1}$$

→ expansion around the reference values from πH and πD

RESULTS for the SIGMA-TERM

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- Apply this linear expansion to the lattice data:



- ⇒ Lattice results clearly at odds with empirical information on the scattering lengths!
- ⇒ lattice must provide scattering lengths with percent accuracy

SUMMARY & OUTLOOK

- Roy-Steiner analysis of $\pi N \rightarrow \pi N$ w/ complete error analysis (first time!)
- Precise determination of the pion-nucleon σ -term: $\sigma_{\pi N} = (59.1 \pm 3.5) \text{ MeV}$
- Consistency check w/o pionic atom data: $\sigma_{\pi N} = (58 \pm 5) \text{ MeV}$
- Open ends:
 - lattice determinations of $\sigma_{\pi N}$ at odds with modern scattering lengths
 - strangeness content $\sim \sigma_s = \langle N | m_s \bar{s}s | N \rangle$
 - FLAG average (doubtful) $\underbrace{\sigma_s = 52.9(7.0) \text{ MeV}}_{N_f=2+1}$ or $\underbrace{\sigma_s = 41.0(8.4) \text{ MeV}}_{N_f=2+1+1}$

Consequences for the proton mass:

About 100 MeV from the Higgs, the rest is gluon field energy

SPARES

PION-NUCLEON SCATTERING

- **s-channel:** $\pi(q) + N(p) \rightarrow \pi(q') + N(p')$
- **t-channel:** $\pi(q) + \pi(-q') \rightarrow \bar{N}(-p) + N(p')$

- **Mandelstam variables:**

$$s = (p + q)^2, t = (p - p')^2, u = (p - q')^2$$

$$s + t + u = 2m_N^2 + 2M_\pi^2, \quad s = W^2$$

- **Isospin structure:**

$$T^{ba}(s, t) = \delta^{ba} T^+(s, t) + i\epsilon_{abc}\tau^c T^-(s, t)$$

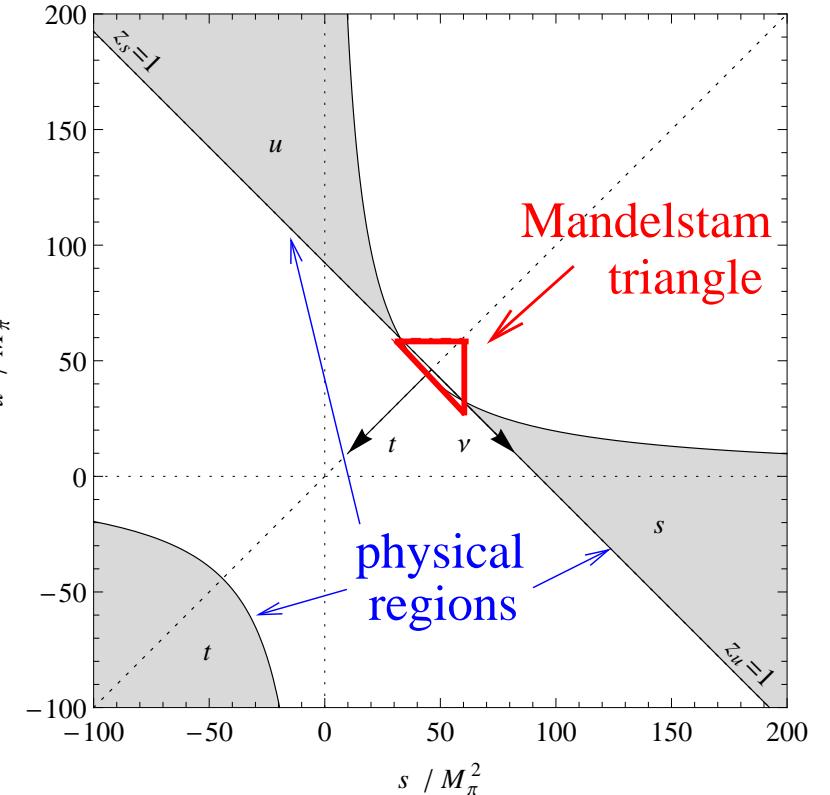
- **Lorentz structure:**

$$8\pi\sqrt{s}T^I(s, t) = \bar{u}(p') \left\{ A^I(s, t) + \frac{1}{2}(\not{q} + \not{q}')B^I(s, t) \right\} u(p), \quad I = +, -$$

$$D^I(s, t) = A^I(s, t) = \nu B^I(s, t), \quad I = 1/2, 3/2$$

- **Crossing:**

$$A^\pm(\nu, t) = \pm A^\pm(-\nu, t), \quad B^\pm(\nu, t) = \mp B^\pm(-\nu, t), \quad \nu = \frac{s - u}{4m_N}$$



PION-NUCLEON SCATTERING continued

- Partial wave projection:

$$X_\ell^I(s) = \int_{-1}^{+1} dz_s P_\ell(z_s) X^I(s, t) \Big|_{t=-2q^2(1-z_s)}, \quad X \in \{A, B\}$$

⇒ partial wave expansion (total isospin I , ang. mom. ℓ , $j = \ell \pm 1/2$):

$$\begin{aligned} f_{\ell\pm}^I(W) &= \frac{1}{16\pi W} \\ &\times \left\{ (E + m) [A_\ell^I(s) + (W - m) B_\ell^I(s)] + (E - m) [-A_{\ell\pm 1}^I(s) + (W + m) B_{\ell\pm 1}^I(s)] \right\} \end{aligned}$$

- MacDowell symmetry: $f_{\ell+}^I(W) = -f_{(\ell+1)-}^I(-W)$ $\forall \ell \geq 0$ MacDowell (1959)

- Low-energy region: only S- and P-waves are relevant

$$f_{0+}^\pm, f_{1+}^\pm, f_{1-}^\pm$$

⇒ low-energy amplitude can eventually be matched to chiral perturbation theory

Büttiker, Fettes, UGM, Steiniger; Ellis, Tang; Becher, Leutwyler, ...

SUBTHRESHOLD EXPANSION

- For the σ -term extraction, the πN amplitude $D = A + \nu B$ is most useful:

$$\bar{D}^+(\nu, t) = D^+(\nu, t) - \frac{g_{\pi N}^2}{m_N} - \nu g_{\pi N}^2 \left(\frac{1}{m_N^2 - s} - \frac{1}{m_N^2 - u} \right)$$

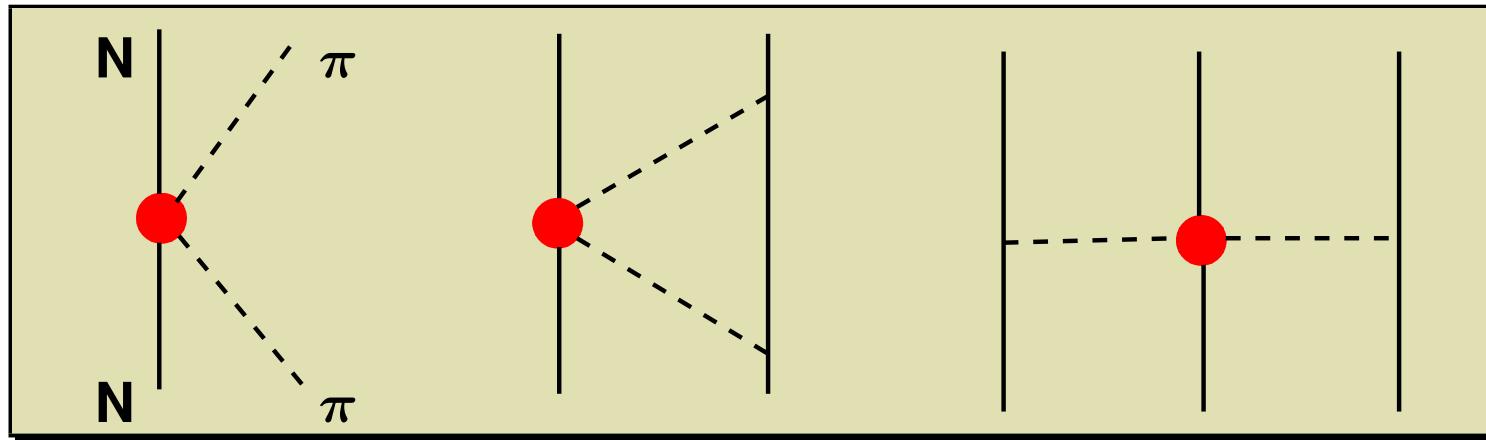
- ★ subtraction of pseudovector Born terms $\rightarrow \bar{D}$
- Subthreshold expansion: expand around $\nu = t = 0$:

$$X(\nu, t) = \sum_{m,n} x_{mn} \nu^{2m} t^n , \quad X \in \left\{ \bar{A}^+, \frac{\bar{A}^-}{\nu}, \frac{\bar{B}^+}{\nu}, \bar{B}^-, \bar{D}^+, \frac{\bar{D}^-}{\nu} \right\}$$

- ★ x_{mn} are the **subthreshold parameters** \rightarrow can be calculated via sum rules
- ★ inside the Mandelstam triangle, scattering amplitudes are real polynomials

PION-NUCLEON SCATTERING & NUCLEAR FORCES

- Low-energy constants (LECs) relate *many* processes
- e.g. the dimension-two LECs c_i in $\pi N, NN, NNN, \dots$



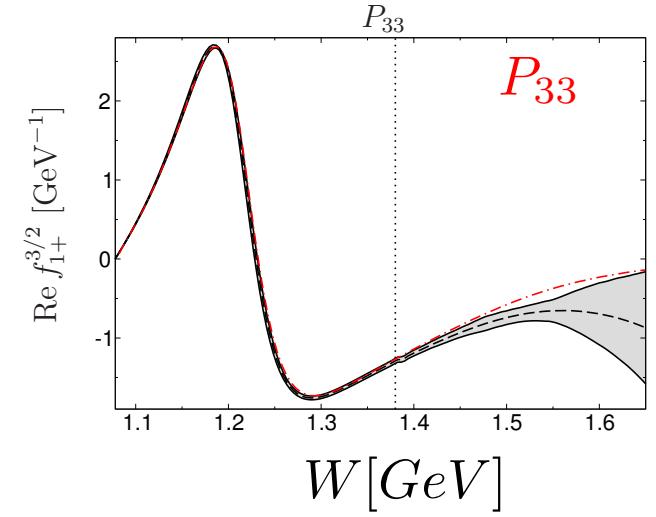
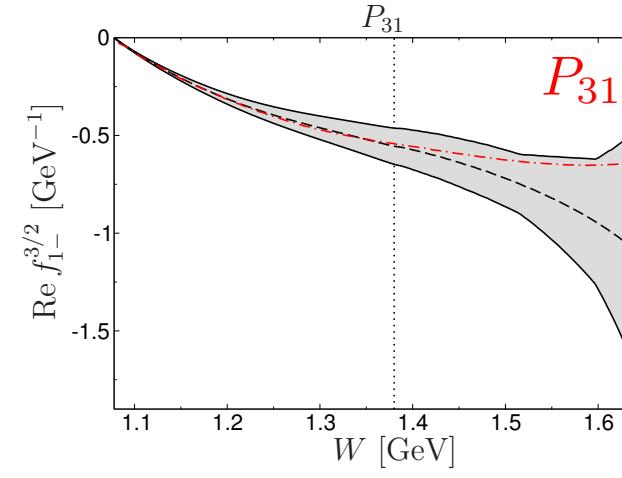
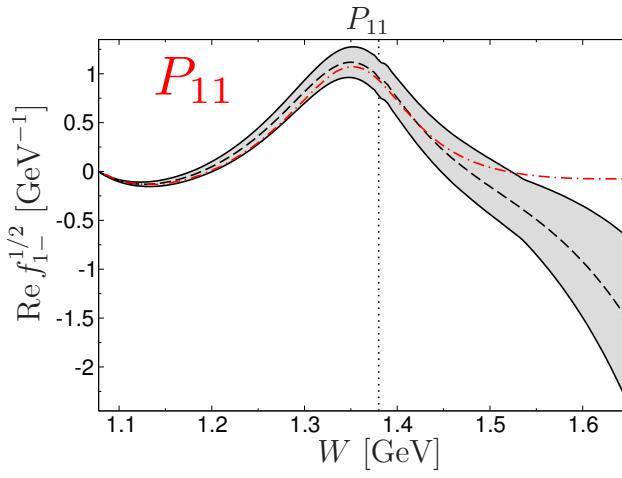
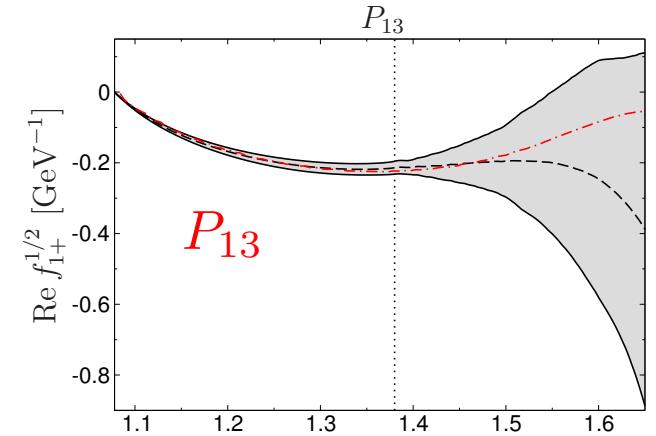
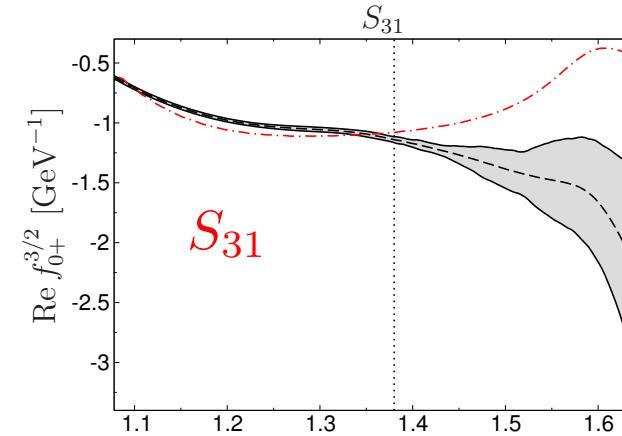
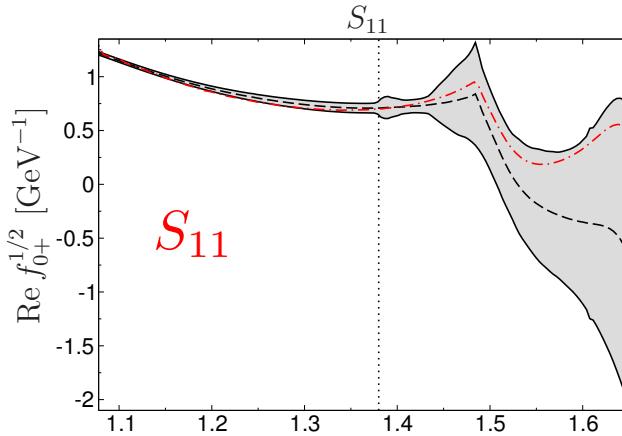
● = operator from $\mathcal{L}_{\pi N}^{(2)} \propto c_i$ ($i = 1, 2, 3, 4$)

Here: • determine the c_i from the purest process $\pi N \rightarrow \pi N$

→ make parameter-free predictions for long-ranged nuclear forces

PHASE SHIFTS II

- Above the matching point (cf SAID/GWU)



THRESHOLD PARAMETERS

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- Threshold parameters from:

$$\text{Re} f_{\ell\pm}^I(s) = q^{2\ell} \left(a_{\ell\pm}^I + b_{\ell\pm}^I q^2 + \dots \right)$$

	RS	KH80
$a_{0+}^{1/2}$	169.8 ± 2.0	173 ± 3
$a_{0+}^{3/2}$	-86.3 ± 1.8	-101 ± 4
$a_{1+}^{1/2}$	-29.4 ± 1.0	-30 ± 2
$a_{1+}^{3/2}$	211.5 ± 2.8	214 ± 2
$a_{1-}^{1/2}$	-70.7 ± 4.1	-81 ± 2
$a_{1-}^{3/2}$	-41.0 ± 1.1	-45 ± 2
$b_{0+}^{1/2}$	-35.2 ± 2.2	-18 ± 12
$b_{0+}^{3/2}$	-49.8 ± 1.1	-58 ± 9

	RS	KH80
a_{0+}^+	-0.9 ± 1.4	-9.7 ± 1.7
a_{0+}^-	85.4 ± 0.9	91.3 ± 1.7
a_{1+}^+	131.2 ± 1.7	132.7 ± 1.3
a_{1+}^-	-80.3 ± 1.1	-81.3 ± 1.0
a_{1-}^+	-50.9 ± 1.9	-56.7 ± 1.3
a_{1-}^-	-9.9 ± 1.2	-11.7 ± 1.0
b_{0+}^+	-45.0 ± 1.0	-44.3 ± 6.7
b_{0+}^-	4.9 ± 0.8	13.3 ± 6.0

- In units of $10^{-3}/M_\pi$ or $10^{-3}/M_\pi^3$, respectively
- As a^+ is very sensitive to isospin breaking and PWAs measure $\pi^\pm p$,
use $(a_{\pi^- p} + a_{\pi^+ p})/2 = (-0.9 \pm 1.4) \cdot 10^{-3}/M_\pi$
- Most striking difference to KH80: S-wave scattering lengths! (here: input)

RESULTS for the SCALAR NUCLEON COUPLINGS

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- WIMP scattering off nuclei sensitive to scalar nucleon couplings:

$$\langle N | m_q \bar{q} q | N \rangle = f_q^N m_N \quad (N = p, n; q = u, d, s)$$

- Include isospin breaking corrections and use $m_u/m_d = 0.46$

$$\Rightarrow f_u^p = (20.8 \pm 1.5) \cdot 10^{-3}, \quad f_d^p = (41.1 \pm 2.8) \cdot 10^{-3}$$

$$f_u^n = (18.9 \pm 1.4) \cdot 10^{-3}, \quad f_d^n = (45.1 \pm 2.7) \cdot 10^{-3}$$

$$\sum_{q=u,\dots,t} f_q^N = \frac{2}{9} + \frac{7}{9} (f_u^N + f_d^N + f_s^N) = 0.305 \pm 0.009$$

- sizeable reduction in uncertainties of $f_{u,d}^N$ due to the precise σ -term
- combination of couplings relevant for Higgs-mediated interactions
- f_s^N from Lattice QCD Junnarkar, Walker-Loud (2013)

RESULTS for the LECs

- Chiral expansion expected to work best at the subthreshold point (polynomial, maximal distance to singularities)
- Express subthreshold parameters in terms of LECs → invert system
- LECs c_i of the dimension two chiral effective πN Lagrangian:

LEC	RS	KGE 2012	UGM 2005
$c_1 \text{ [GeV}^{-1}]$	-1.11 ± 0.03	$-1.13 \dots - 0.75$	$-0.9^{+0.2}_{-0.5}$
$c_2 \text{ [GeV}^{-1}]$	3.13 ± 0.03	$3.49 \dots 3.69$	3.3 ± 0.2
$c_3 \text{ [GeV}^{-1}]$	-5.61 ± 0.06	$-5.51 \dots - 4.77$	$-4.7^{+1.2}_{-1.0}$
$c_4 \text{ [GeV}^{-1}]$	4.26 ± 0.04	$3.34 \dots 3.71$	$-3.5^{+0.5}_{-0.2}$

Krebs, Gasparyan, Epelbaum, Phys. Rev. C85 (2012) 054006
 UGM, PoS LAT2005 (2006) 009

- also results for pertinent dimension three and four LECs

t -CHANNEL MO PROBLEM

- One-channel MO problem with finite matching point t_m

$$f(t) = \Delta(t) + \frac{1}{\pi} \int_{4M_\pi^2}^{t_m} dt' \frac{T(t')^\star f(t')}{t' - t} + \frac{1}{\pi} \int_{t_m}^\infty dt' \frac{\text{Im } f(t')}{t' - t}$$

→ $\Delta(t)$: pole terms, s-channel imag. parts, other t -channel PWs

→ solve for $f(t)$ in $4M_\pi^2 \leq t \leq t_m$ requires

- $\text{Im } f(t)$ for $t \geq t_m$
- $T(t)$ for $4M_\pi^2 \leq t \leq t_m$

- Solution via once-subtracted Omnès function (w/ $\Omega(0) = 1$):

$$\Omega(t) = \exp \left\{ \frac{t}{\pi} \int_{4M_\pi^2}^{t_m} \frac{dt'}{t'} \frac{\delta(t')}{t' - t} \right\} = |\Omega(t)| \exp \{ i\delta(t)\theta(t - 4M_\pi^2)\theta(t_m - t) \}$$

Flavor decomposition of the σ term

Severt, Gegelia, UGM, JHEP **1903** (2019) 202 [arXiv:1902.10508]

WHAT DO WE KNOW ABOUT σ_0 ?

- Chiral expansion: $\sigma_0 = \sigma_0^{(2)} + \sigma_0^{(3)} + \sigma_0^{(4)} + \dots$
- Lowest order: $\sigma_0^{(2)} = \frac{1}{2} \left(\frac{M_\pi^2}{M_K^2 - M_\pi^2} \right) (m_\Xi + m_\Sigma - 2m_N) \simeq 27 \text{ MeV}$
- Higher orders: $\sigma_0 = (35 \pm 5) \text{ MeV}$ cutoff chiral Lagrangian at one loop
Gasser (1981)
 $\sigma_0^{(4)} = (36 \pm 7) \text{ MeV}$ heavy baryon CHPT at one loop
Borasoy, UGM (1997)
 $\sigma_0^{(3)} = 46 - 89 \text{ MeV}$ heavy baryon/cov. CHPT w/ or w/o Delta
Alarcon et al. (2014)
- Situation is confusing / inconsistent

\Rightarrow need a complete $O(p^4)$ calculation w/ and w/o the decuplet
 in both the heavy baryon and a covariant (EOMS) approach

NEW CALCULATION of σ_0

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- Use chiral Lagrangians of octet baryons plus Goldstone boson octet (plus decuplet)
- Use both the covariant EOMS scheme as well as the heavy baryon approach

- Use the Feynman-Hellmann theorem:

$$\hat{m} \left(\frac{\partial m_N}{\partial \hat{m}} \right) = \sigma_{\pi N}$$

[→ extra slide]

- with $\frac{\partial m_N}{\partial \hat{m}} = \left(\frac{\partial m_N}{\partial M_\pi} \right) \frac{\partial M_\pi}{\partial \hat{m}} + \left(\frac{\partial m_N}{\partial M_K} \right) \frac{\partial M_K}{\partial \hat{m}} + \left(\frac{\partial m_N}{\partial M_\eta} \right) \frac{\partial M_\eta}{\partial \hat{m}}$

- Baryon masses

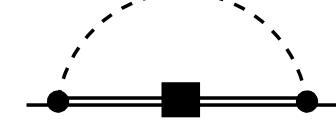
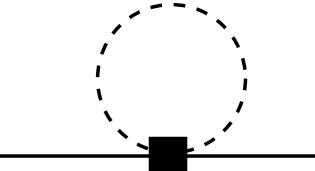
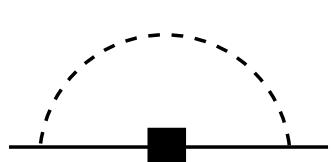
fourth order



$O(p^2) + O(p^4)$ trees



$O(p^3)$ loops



$O(p^4)$ loops

FEYNMAN-HELLMANN THEOREM

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Hellmann (1937), Feynman (1939)

- QM Hamiltonian $H(\lambda)$, depending on some external parameter λ

$$H(\lambda)|\psi(\lambda)\rangle = E(\lambda)|\psi(\lambda)\rangle, \quad \langle\psi(\lambda)|\psi(\lambda)\rangle = 1$$

- Differentiation w.r.t. λ gives:

$$\begin{aligned} \frac{dE(\lambda)}{d\lambda} &= \frac{d}{d\lambda} \langle\psi(\lambda)|H(\lambda)|\psi(\lambda)\rangle \\ &= \left\langle\psi(\lambda) \left| \frac{dH(\lambda)}{d\lambda} \right| \psi(\lambda)\right\rangle + \left\langle\frac{\psi(\lambda)}{d\lambda} \left| H(\lambda) \right| \psi(\lambda)\right\rangle + \left\langle\psi(\lambda) \left| H(\lambda) \right| \frac{d\psi(\lambda)}{d\lambda}\right\rangle \\ &= \left\langle\psi(\lambda) \left| \frac{dH(\lambda)}{d\lambda} \right| \psi(\lambda)\right\rangle + E(\lambda) \frac{d}{d\lambda} \underbrace{\langle\psi(\lambda)|\psi(\lambda)\rangle}_{=1} \\ &= \left\langle\psi(\lambda) \left| \frac{dH(\lambda)}{d\lambda} \right| \psi(\lambda)\right\rangle \end{aligned}$$

- In QCD, identify the quark masses with the external parameter λ

$$\frac{dM_H}{dm_f} = \langle H|\bar{\psi}_f\psi_f|H\rangle \quad \text{make scale-invariant} \rightarrow \boxed{\sigma_H^f = m_f \langle H|\bar{\psi}_f\psi_f|H\rangle}$$

[beware of normalization of the hadron state!]

CHIRAL LAGRANGIAN

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- Consider first the meson-baryon system:

$$\mathcal{L}_{\phi B} = \mathcal{L}_{\phi B}^{(1)} + \mathcal{L}_{\phi B}^{(2)} + \mathcal{L}_{\phi B}^{(3)} + \mathcal{L}_{\phi B}^{(3)}$$

- Leading order $\mathcal{O}(p)$:

$$\mathcal{L}_{\phi B}^{(1)} = \text{Tr} (\bar{B} (iD - m_0) B) + \frac{D}{2} \text{Tr} (\bar{B} \gamma^\mu \gamma_5 \{u_\mu, B\}) + \frac{F}{2} \text{Tr} (\bar{B} \gamma^\mu \gamma_5 [u_\mu, B])$$

- LECs F & D from hyperon decays, m_0 the first undetermined LEC

- NLO $\mathcal{O}(p^2)$ (symmetry breaking and interaction terms)

$$\mathcal{L}_{\phi B}^{(2)} = \mathcal{L}_{\phi B}^{(2, \text{sb})} + \mathcal{L}_{\phi B}^{(2, \text{int})}$$

$$\mathcal{L}_{\phi B}^{(2, \text{sb})} = b_0 \text{Tr}(\chi_+) \text{Tr}(\bar{B} B) + b_D \text{Tr}(\bar{B} \{\chi_+, B\}) + b_F \text{Tr}(\bar{B} [\chi_+, B])$$

$$\begin{aligned} \mathcal{L}_{\phi B}^{(2, \text{int})} &= b_1 \text{Tr} (\bar{B} [u_\mu, [u^\mu, B]]) + \dots \\ &+ i b_8 \left(\text{Tr} (\bar{B} \gamma_\mu D_\nu B) - \text{Tr} (\bar{B} \overleftrightarrow{D}_\nu \gamma_\mu B) \right) \text{Tr} (u^\mu u^\nu) \end{aligned}$$

- LECs $b_{0,D,F}$ in second order masses, LECs $b_{1,\dots,8}$ in fourth order tadpoles

- NNLO $\mathcal{O}(p^3)$ does not contribute to the masses at one loop

- NNNLO $\mathcal{O}(p^4)$:

$$\begin{aligned} \mathcal{L}_{\phi B}^{(4)} = & d_1 \text{Tr}(\bar{B}[\chi_+, [\chi_+, B]]) + d_2 \text{Tr}(\bar{B}[\chi_+, \{\chi_+, B\}]) + d_3 \text{Tr}(\bar{B}\{\chi_+, \{\chi_+, B\}\}) \\ & + d_4 \text{Tr}(\bar{B}\chi_+) \text{Tr}(\chi_+ B) + d_5 \text{Tr}(\bar{B}[\chi_+, B]) \text{Tr}(\chi_+) + d_7 \text{Tr}(\bar{B}B) [\text{Tr}(\chi_+)]^2 \\ & + d_8 \text{Tr}(\bar{B}B) \text{Tr}(\chi_+^2) \end{aligned}$$

- LECs $d_{1,\dots,5,7,8}$ in fourth order trees
- Inclusion of the decuplet \rightarrow extra slide
- Up-to-order $\mathcal{O}(p^3)$, less LECs than fit parameter (baryon masses)
- At complete one-loop, *too many* LECs
 - \rightarrow ultimately, use LQCD (but at present this is questionable)
 - \rightarrow constrain from matching to SU(2) (if possible) [\rightarrow extra slide]
 - \rightarrow use Bayesian analysis w/ central values around 0 and given uncertainty

CHIRAL LAGRANGIAN: INCLUSION of the DECUPLLET

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- why decuplet? a) small mass difference and b) large coupling to the πN system

\rightarrow count $m_D - m_B$ as small parameter

Hemmert, Holstein, Kambor (1998)

- Use a Rarita-Schwinger field T_μ^{abc} : $T_\mu^{111} = \Delta_\mu^{++}, \dots, T_\mu^{333} = \Omega_\mu^-$

- LO Lagrangians: $\mathcal{L}_D^{(1)} = \bar{T}_\mu^{abc} (i\gamma^{\mu\nu\rho} D_\rho - m_D \gamma^{\mu\nu}) T_{\nu, abc}$

$$\mathcal{L}_{DB\phi}^{(1)} = \frac{\mathcal{C}}{2} \left\{ \bar{T}_\mu^{abc} \Theta^{\mu\nu}(z) (u_\nu)_a^i B_b^j \epsilon_{cij} + \bar{B}_j^b (u_\nu)_i^a \Theta^{\nu\mu}(z) T_{\mu, abc} \epsilon^{cij} \right\}$$

$$\Theta^{\mu\nu}(z) = g^{\mu\nu} (z + \tfrac{1}{2}) \gamma^\mu \gamma^\nu$$

- Off-shell parameter z not observable ($z = 1/2$) and LEC \mathcal{C} from $\Gamma(\Delta \rightarrow N\pi)$

Tang, Ellis, Phys. Lett. B **387** (1996) 9; Krebs, Epelbaum, UGM, Phys. Lett. B **683** (2010) 222

- NLO Lagrangians:

$$\mathcal{L}_D^{(2, \text{sb})} = \frac{t_0}{2} \text{Tr}(\chi_+) \bar{T}_\mu^{abc} g^{\mu\nu} T_{\nu, abc} + \frac{t_D}{2} \bar{T}_\mu^{abc} g^{\mu\nu} (\chi_+, T_\nu)_{abc}$$

- the LECs t_0 and t_D can be extracted from the $\mathcal{O}(p^3)$ decuplet masses

Camalich, Geng, Vacas, Phys. Rev. D **82** (2010) 074504

MATCHING

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- Step 1: Integrate out the strange quark from the theory
- Step 2: Match the SU(3) LECs to the SU(2) counterparts
→ constraints (depend on the order!)
- The simplest matching relation: $g_A = F + D + \mathcal{O}(\hat{M}_K^3)$ [$\hat{M}_K^2 = m_s B_0$]
- Dynamical LECs: Frink, UGM, JHEP **0407** (2004) 028

$$\frac{1}{4}c_2 + c_3 = b_1 + b_2 + b_3 + 2b_4 - \frac{1}{2}m_B(b_5 + b_6 + b_7 + 2b_8)$$

$$c_1 = b_0 + \frac{1}{2}(b_D + b_F) + \text{many more terms}$$

$$-4\bar{e}_1 + \frac{3}{8} \frac{1}{(4\pi F_\pi)^2} c_2 = \text{many terms}$$

- Left side: use the precise values from the RS analysis;
right side: impose as constraint → there still remain undetermined LECs!

RESULTS for σ_0

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- Third and fourth order results for σ_0 [MeV]:

	HB	HB + decuplet	EOMS	EOMS + decuplet
$\mathcal{O}(p^3)$	57.9(0.2)(17.0)	88.6(0.2)(34.0)	46.4(0.2)(10.4)	57.6(0.2)(17.0)
$\mathcal{O}(p^4)$, unc.	64.1(31.7)(9.3)	64.0(31.7)(18.7)	51.8(31.4)(5.7)	61.8(31.4)(9.3)
$\mathcal{O}(p^4)$, con.	58.0	60.4	42.8	60.9

- First error: Uncertainty within the order; second error: neglect of higher orders

$$(\Delta X)_{\text{theo}}^{(N)} = \max(|X^{(N_{\text{LO}})}|Q^{N-N_{\text{LO}}+1}, \{|X^{(k)} - X^{(j)}|Q^{n-j}\})$$

$$Q = \max\left(\frac{p}{\Lambda_\chi}, \frac{M_\phi}{\Lambda_\chi}\right), \quad Q = \max\left(\frac{p}{\Lambda_\chi}, \frac{M_\phi}{\Lambda_\chi}, \frac{\Delta}{\Lambda_\chi}\right)$$

Epelbaum, Krebs, UGM, Eur. Phys. J. A 51 (2015) 53

- Results converge to $\sigma_0 \simeq 60$ MeV

$y \simeq 0 \rightarrow$ little strangeness in the nucleon

