Origin and Resummation of Threshold Logarithms in the Lattice QCD Calculations of PDFs

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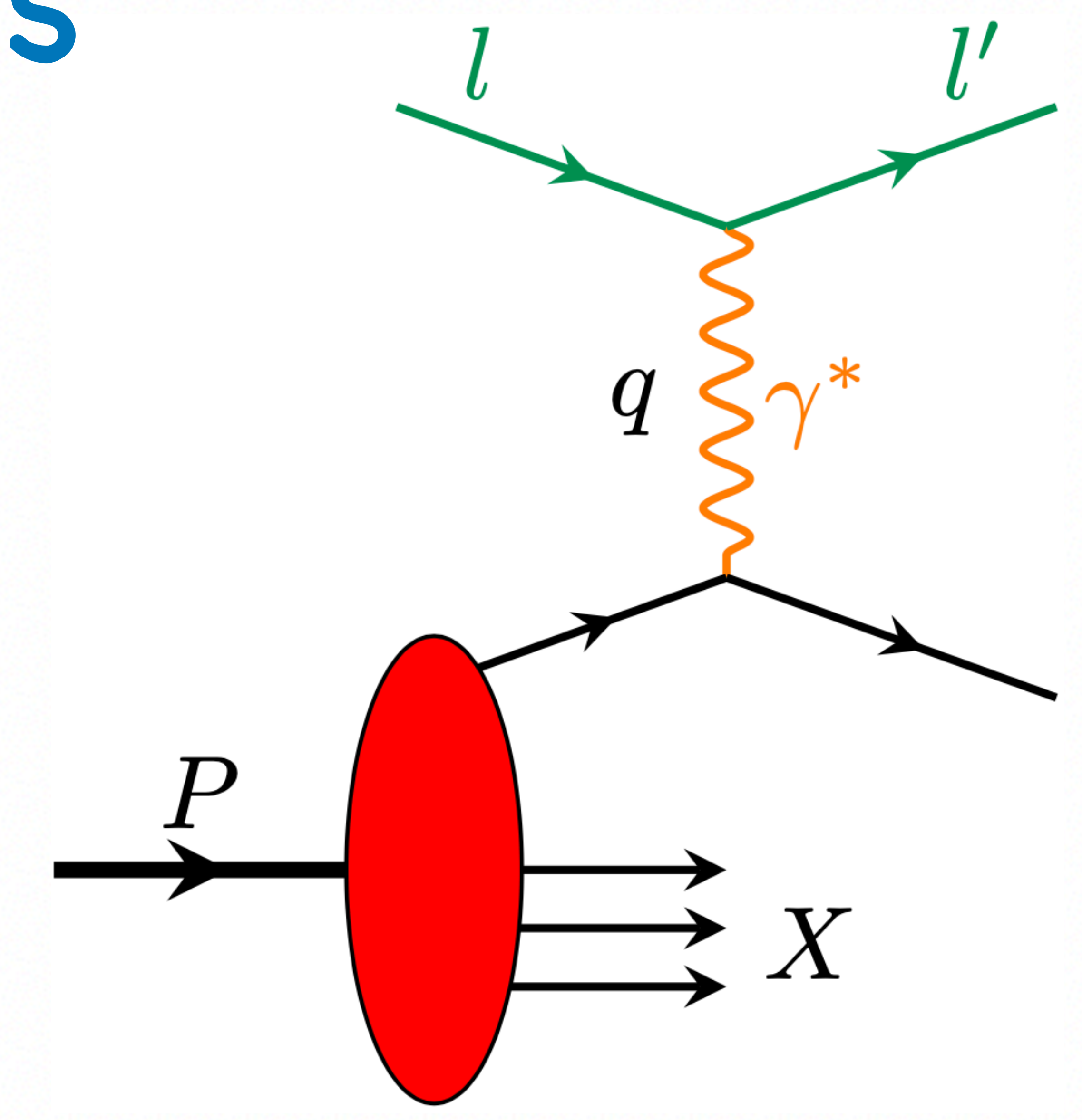
Based on **Phys.Rev.D 103 (2021) 9, 094504** with Kyle Lee, Swagato Mukherjee, Charles Shugert, and Yong Zhao

LaMET 2022, Dec 1 – 3, Chicago

Parton distribution functions

DIS



Non-perturbative PDFs

$$\sigma = \sum_i f_i(x, Q^2) \circledast \sigma \{eq_i(xP) \rightarrow eq_i(xP + q)\}$$

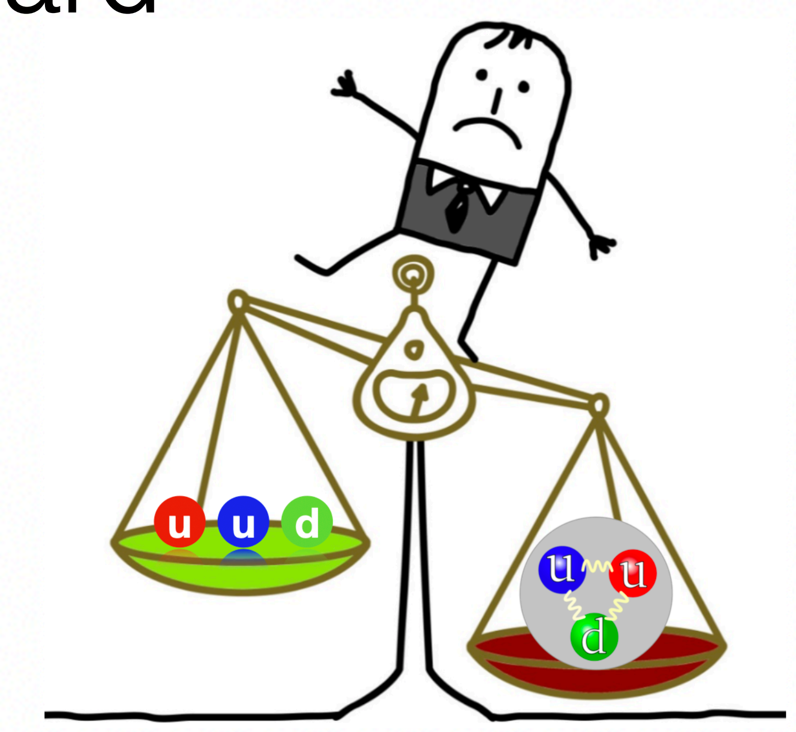
Perturbative parton process

Hadron Structure and Tomography:

- How hadrons are built.
- Mass and spin decomposition of hadron.

High-energy phenomenology:

- Standard Model backgrounds.
- Higgs physics and search for physics beyond the Standard Model.



Large momentum effective theory (LaMET)

Quasi-PDFs Factorization

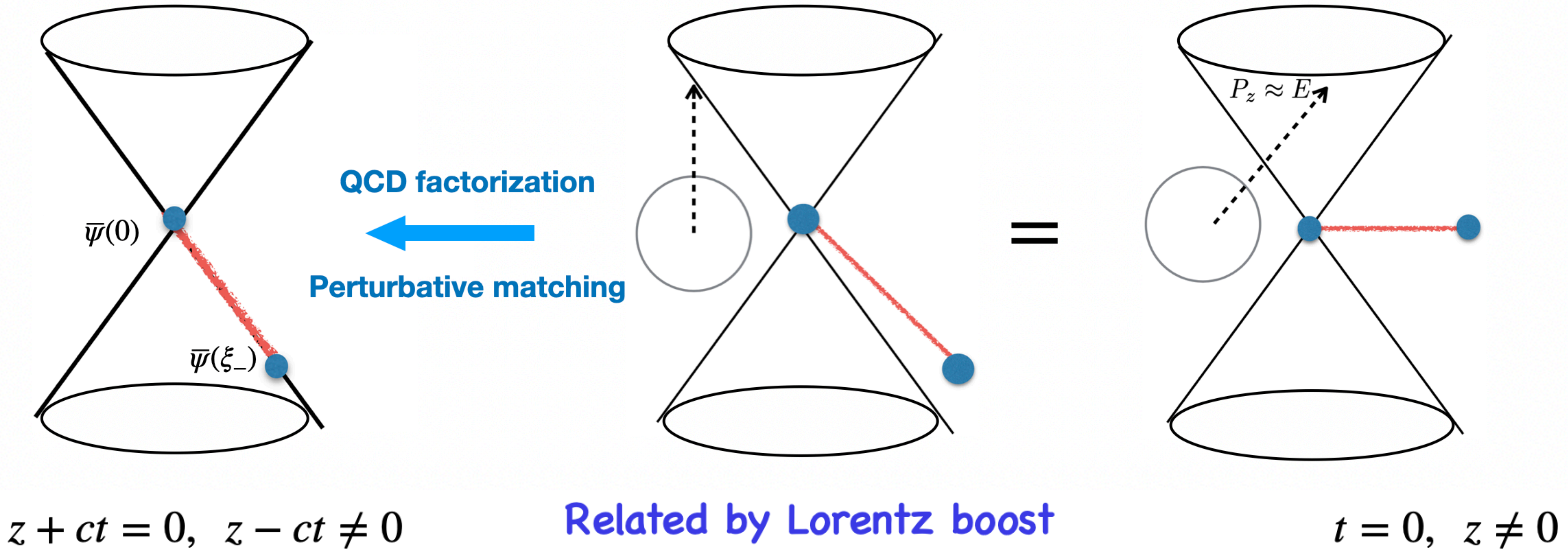
IR must cancel

$$\tilde{q}(x, P_z) = q(x,\mu) + \alpha_s(\mu)(\tilde{q}^{(1)}(x, P_z, \mu) - q^{(1)}(x,\mu))$$

$$= \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP_z}\right) q(y,\mu) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{x^2P_z^2}, \frac{\Lambda_{QCD}^2}{(1-x)P_z^2}\right)$$

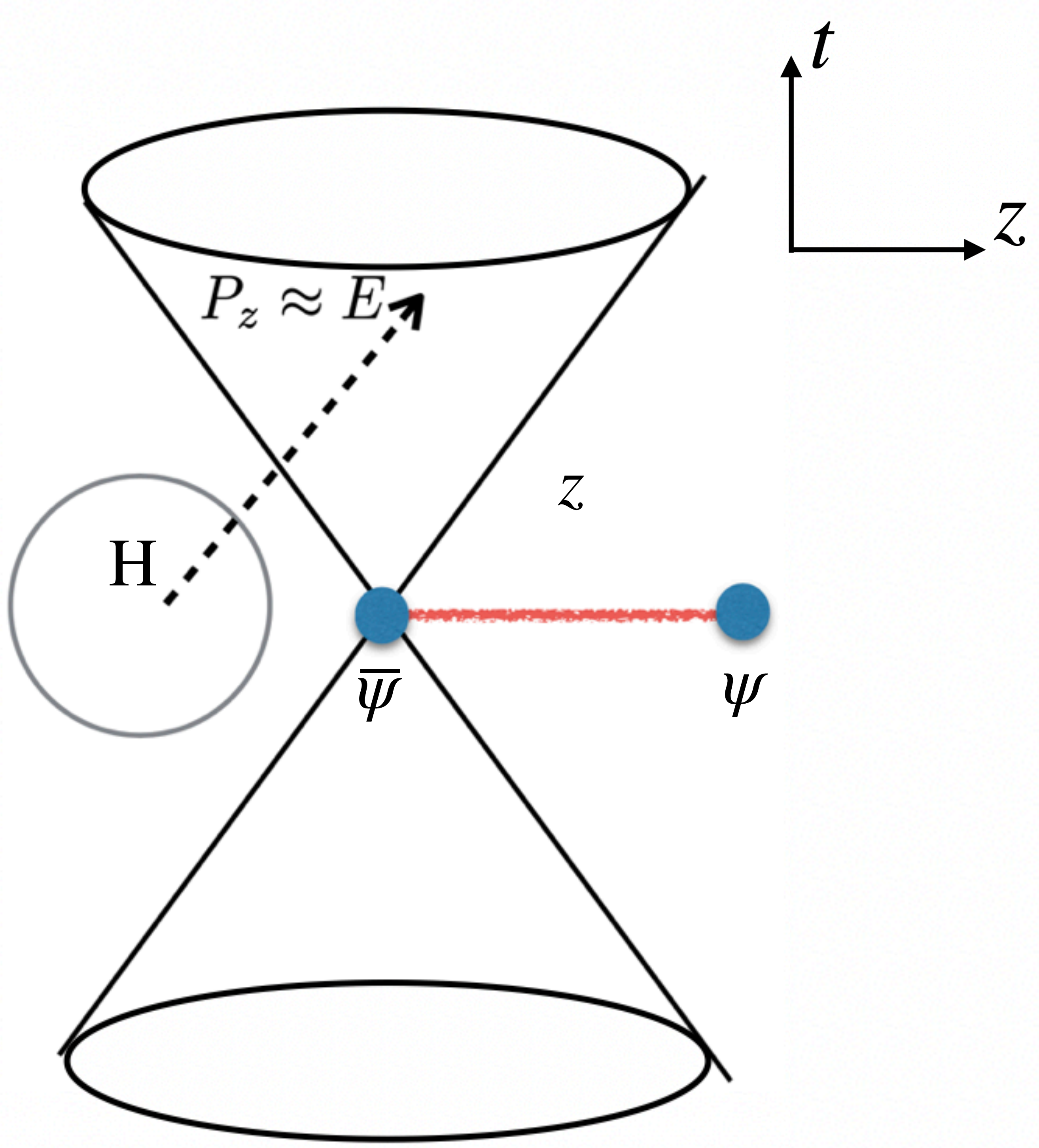
- X. Ji, PRL 110 (2013); SCPMA57 (2014);
- X. Xiong, X. Ji, et al, PRD 90 (2014);
- Y.-Q. Ma, et al, PRD98 (2018), PRL 120 (2018);
- T. Izubuchi, X. Ji, et al PRD98 (2018).
- X. Ji, Y. Zhao, et al, RMP 93 (2021).

large P_z is essential



Pseudo PDF / short distance factorization

$t = 0, \qquad z \neq 0$



quasi-PDF matrix elements

$$\langle P \,|\, \tilde{O}_\Gamma(z,\epsilon) \,|\, P \rangle$$

$$\tilde{O}_\Gamma(z,\epsilon) = \overline{\psi}(0)\Gamma W_{\hat{z}}(0,z)\psi(z)$$

Short-distance factorization:

- V. Braun et al., EPJC 55 (2008)
- A. V. Radyushkin et al., PRD 96 (2017)
- Y. Ma et al., PRL 120 (2018)
- T. Izubuchi et al., PRD 98 (2018)

$$h^R(\lambda, z^2, \mu) = h^R(z, P_z, \mu)$$

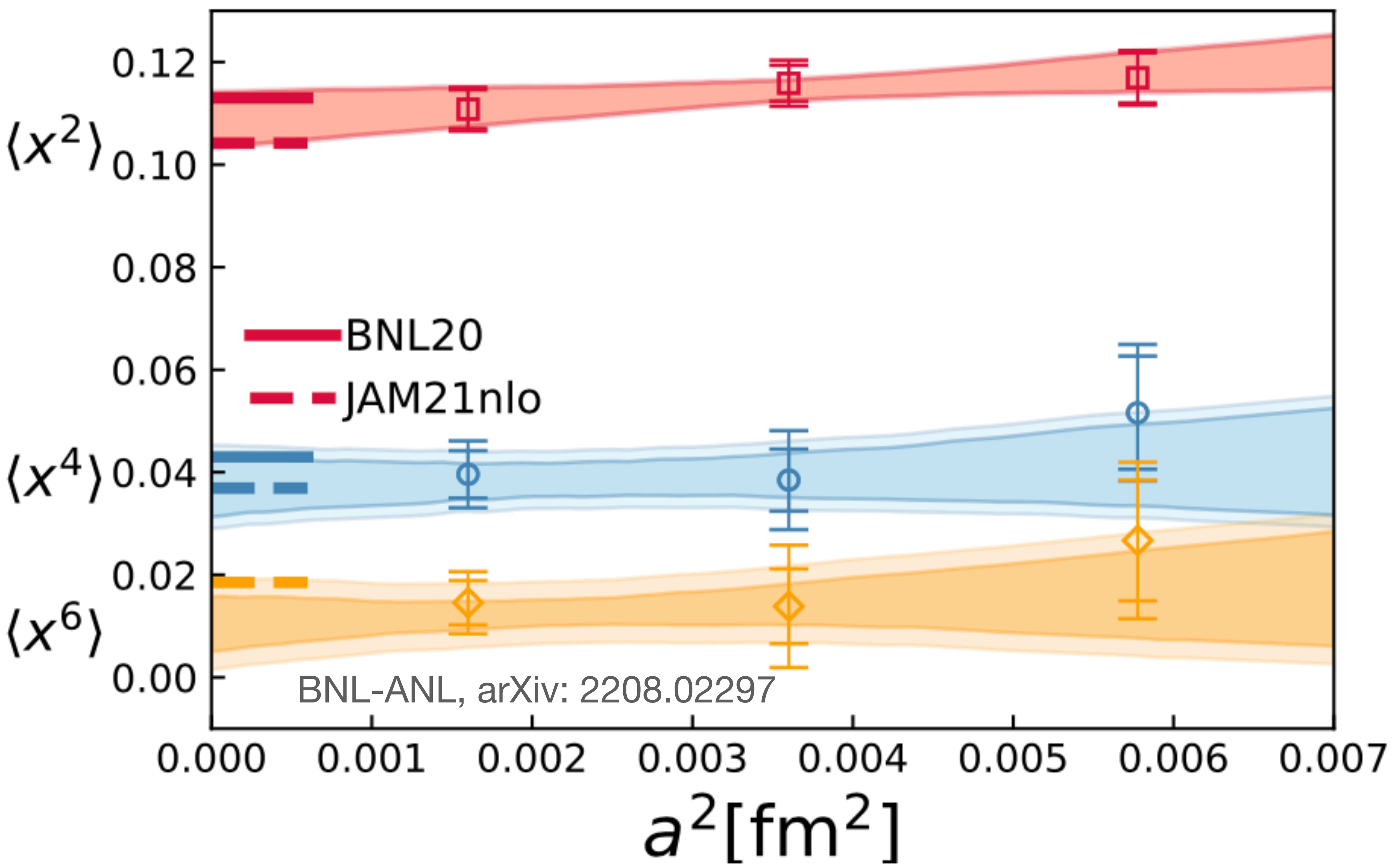
$$= \underbrace{\int_{-1}^{1} d\alpha \mathscr{C}(\alpha, \mu^2 z^2)}_{\text{Perturbative kernel}} \int_{-1}^{1} dy e^{-iy\alpha\lambda} q(y,\mu) + \mathcal{O}(z^2\Lambda_{QCD}^2)$$

$$\lambda = zP_z$$

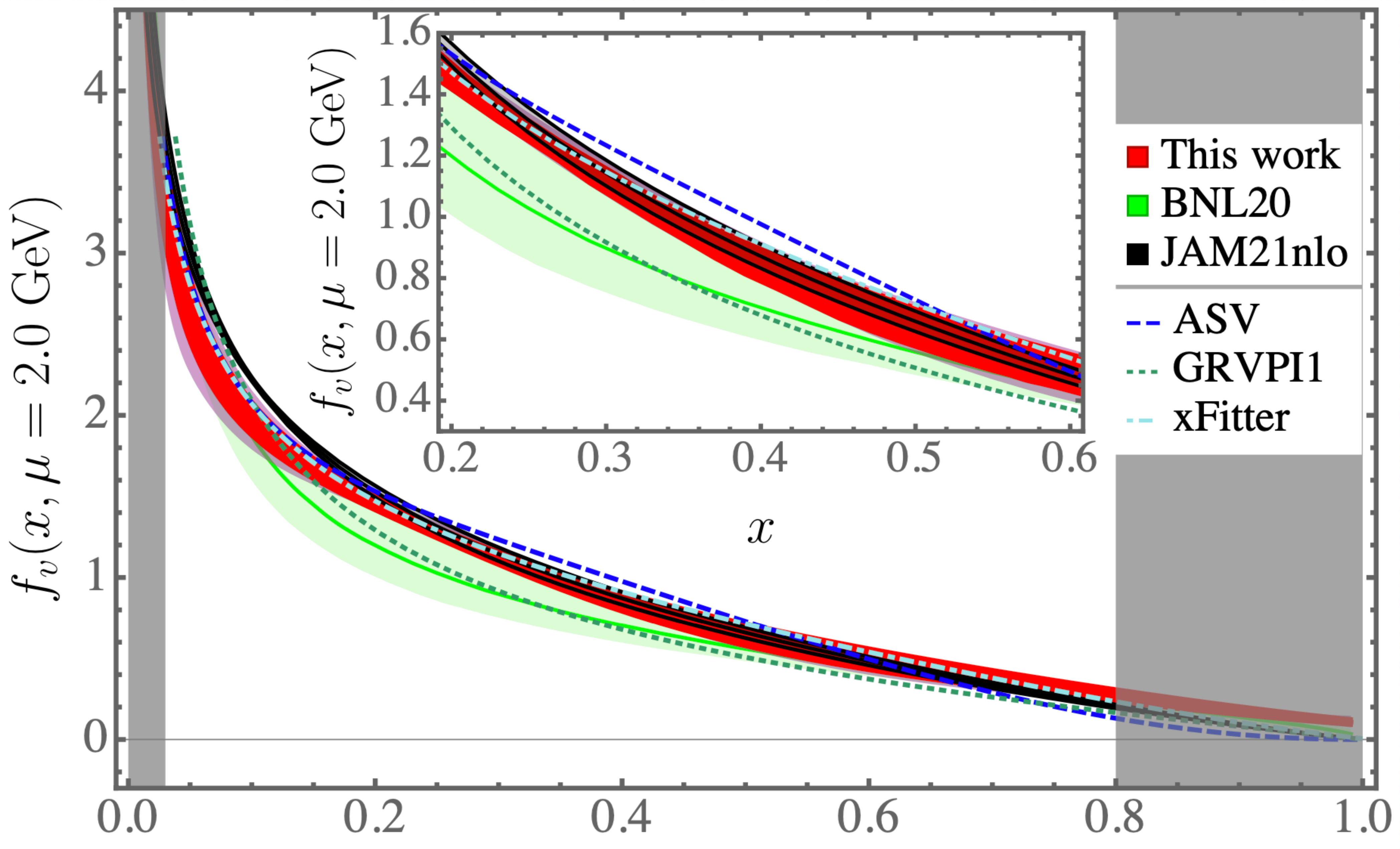
- The perturbative matching is valid in short range of z.
- The information that lattice data contains is limited by the range of finite $\lambda = zP_z$.

Pion valence quark PDF: fixed order

Moments of pion valence quark PDF



Pion valence quark PDF

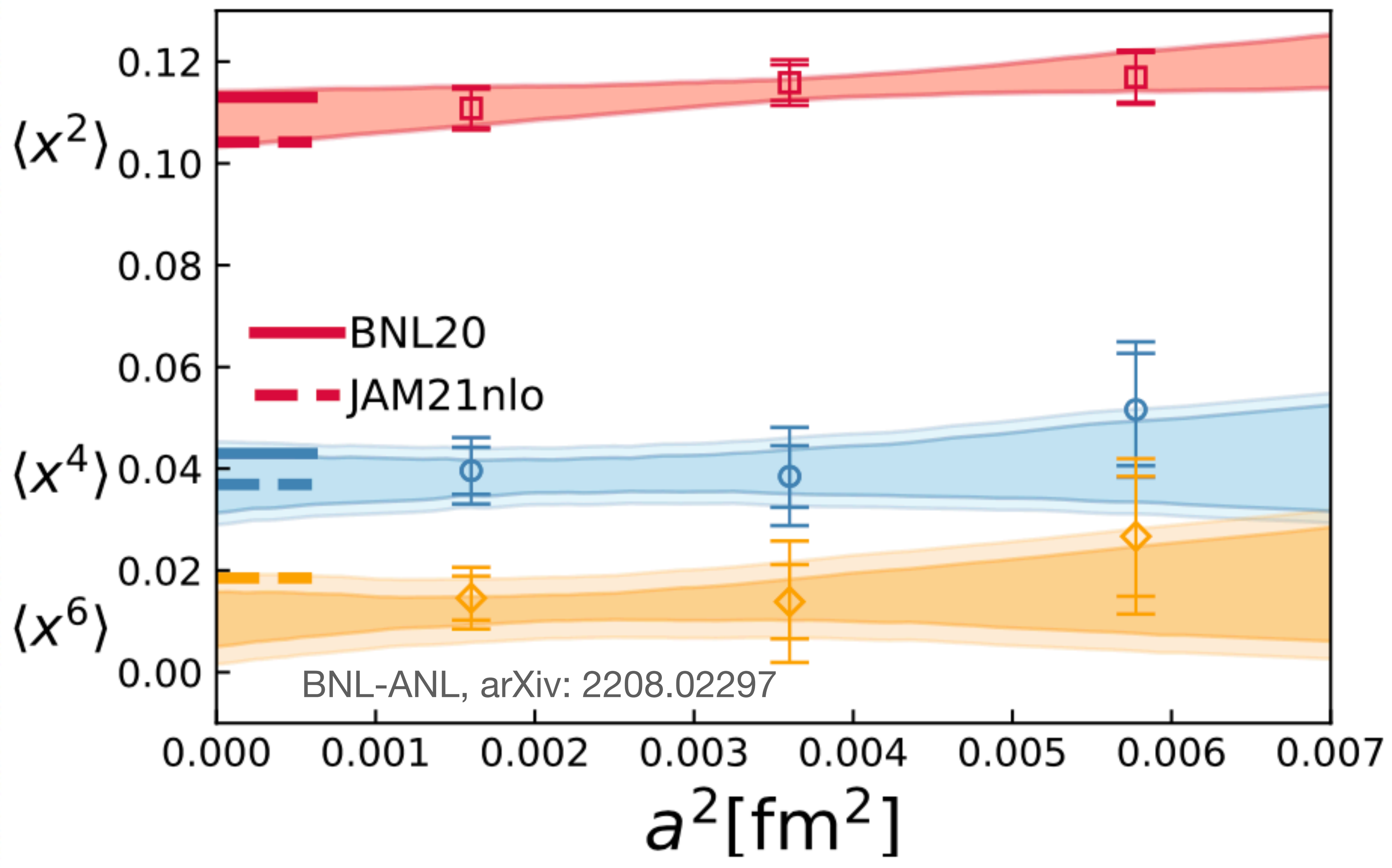


- BNL-ANL, PRL 128 (2022) 14, 142003

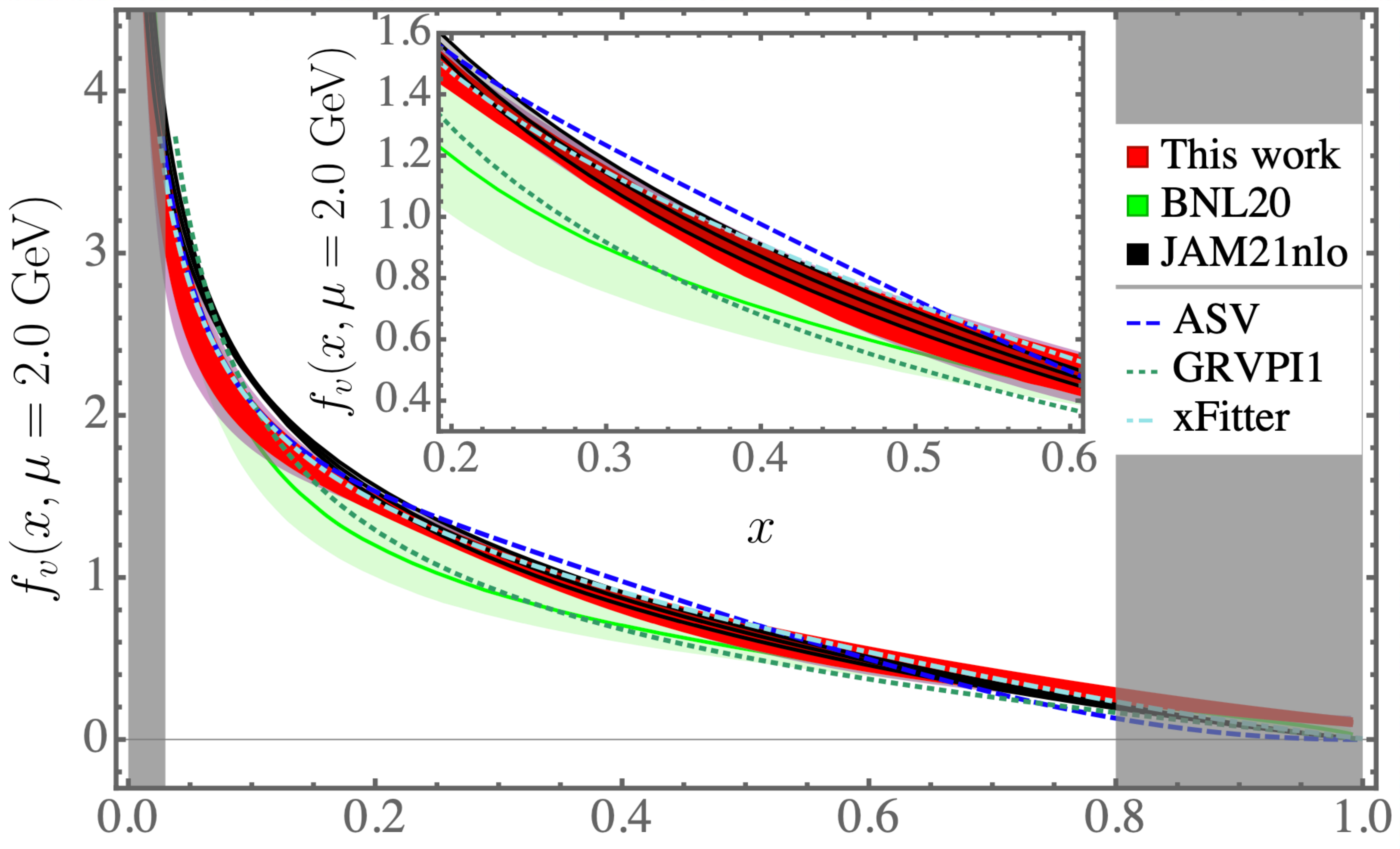
- Lattice prediction of pion valence PDF show good agreement with most recent Global analysis from JAM, xFitter.

Pion valence quark PDF: fixed order

Moments of pion valence quark PDF



Pion valence quark PDF



- BNL-ANL, PRL 128 (2022) 14, 142003

Improvement:

- Higher statistics.
- Lattice artifacts: smaller lattice spacing, chiral fermion calculations.
- Power correction: larger momentum.
- Perturbative matching: higher order, resummation.
- …

Large momentum effective theory (LaMET)

Quasi-PDFs Factorization:

$$\tilde{q}(x, P_z) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP_z}\right) q(y, \mu) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{x^2 P_z^2}, \frac{\Lambda_{QCD}^2}{(1-x)P_z^2}\right)$$

One-loop matching:

$$C^{(1)}\left(\xi, \frac{\mu}{|y|P_z}\right) = \frac{\alpha_s C_F}{2\pi}\delta(1-\xi)\left[\frac{3}{2}\boxed{\ln\frac{\mu^2}{4y^2P_z^2}} - \frac{5}{2}\right]$$

$$+\frac{\alpha_s C_F}{2\pi}\begin{cases} \left(\dfrac{1+\xi^2}{1-\xi}\ln\dfrac{\xi}{\xi-1} + 1\right)_+ & \xi > 1 \\ \dfrac{1+\xi^2}{1-\xi}\left[\boxed{-\ln\dfrac{\mu^2}{4y^2P_z^2}} - \ln\dfrac{\xi}{1-\xi} - 1\right]_+ & 0 < \xi < 1 \\ \left(-\dfrac{1+\xi^2}{1-\xi}\ln\dfrac{-\xi}{1-\xi} - 1\right)_+ & \xi < 0 \end{cases}$$

DGLAP evolution

- X. Ji, Y. Zhao, et al, RMP 93 (2021).
- Y. Su et al, arXiv: 2209.01236

$$\frac{dC^{(1)}\left(\xi, \frac{\mu}{yP_z}\right)}{d\ln(yP_z)} = \frac{\alpha_s}{\pi}\left[P_{qq}^{(0)}(\xi) - \frac{3}{2}(1-\xi)\right]$$

Large momentum effective theory (LaMET)

Quasi-PDFs Factorization:

$$\tilde{q}(x, P_z) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP_z}\right) q(y, \mu) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{x^2 P_z^2}, \frac{\Lambda_{QCD}^2}{(1-x)P_z^2}\right)$$

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$$+\frac{\alpha_s C_F}{2\pi}\begin{cases} \left(\dfrac{1+\xi^2}{1-\xi}\ln\dfrac{\xi}{\xi-1} + 1\right)_+ & \xi > 1 \\ \dfrac{1+\xi^2}{1-\xi}\left[-\ln\dfrac{\mu^2}{4y^2P_z^2} - \ln\dfrac{\xi}{1-\xi} - 1\right]_+ & 0 < \xi < 1 \\ \left(-\dfrac{1+\xi^2}{1-\xi}\ln\dfrac{-\xi}{1-\xi} - 1\right)_+ & \xi < 0 \end{cases}$$

Threshold logarithms

- X. Gao, et al, Phys.Rev.D 103 (2021) 9

$$\lim_{\substack{x,y\to 1\\ \xi\to 1}} C^{(1)}\left(\xi, \frac{\mu}{|y|p^z}\right)$$
$$\sim \frac{\alpha_s C_F}{2\pi}\left[\underbrace{\frac{2\ln|1-\xi|}{|1-\xi|} - \frac{2}{(1-\xi)}\ln\frac{\mu^2}{4p_z^2}}_{\dfrac{2}{|1-\xi|}\ln\dfrac{|1-\xi|P_z^2}{\mu^2}} + \frac{3}{2|1-\xi|}\right]_+$$

$\xi \to 1$ approaching Landau pole

The pseudo distribution

Short-distance Factorization:

$$h^R(\lambda, z^2, \mu) = \int_{-1}^{1} d\alpha \mathscr{C}(\alpha, \mu^2 z^2) \int_{-1}^{1} dy e^{-iy\alpha\lambda} q(y, \mu) + \mathcal{O}(z^2 \Lambda_{QCD}^2)$$

One-loop matching:

$$\mathscr{C}^{(1)}(\alpha, z^2\mu^2) = \delta(1-\alpha)\frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} + \frac{5}{2}\right]$$

$$+\frac{\alpha_s C_F}{2\pi}\{\left(\frac{1+\alpha^2}{1-\alpha}\right)_+ \left[-\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} - 1\right]$$

$$-\left(\frac{4\ln(1-\alpha)}{1-\alpha}\right)_+ + 2(1-\alpha)_+\}\theta(\alpha)\theta(1-\alpha)$$

DGLAP evolution

- A. V. Radyushkin, Phys.Lett.B 781 (2018).
- X. Ji, Y. Zhao, et al, RMP 93 (2021).
- Y. Su et al, arXiv: 2209.01236

$$\frac{d\mathscr{C}(\alpha, \mu^2 z^2)}{d\ln z^2} = \frac{\alpha_s}{2\pi}\left[-P_{qq}^{(0)}(\alpha) - \frac{3}{2}(1-\alpha)\right]$$

The pseudo distribution

Short-distance Factorization:

$$h^R(\lambda, z^2, \mu) = \int_{-1}^{1} d\alpha \mathscr{C}(\alpha, \mu^2 z^2) \int_{-1}^{1} dy e^{-iy\alpha\lambda} q(y, \mu) + \mathcal{O}(z^2 \Lambda_{QCD}^2)$$

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$$+\frac{\alpha_s C_F}{2\pi}\{\left(\frac{1+\alpha^2}{1-\alpha}\right)_+ \left[-\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} - 1\right]$$
$$-\left(\frac{4\ln(1-\alpha)}{1-\alpha}\right)_+ + 2(1-\alpha)_+\}\theta(\alpha)\theta(1-\alpha)$$

Threshold logarithms

- X. Gao, et al, Phys.Rev.D 103 (2021) 9

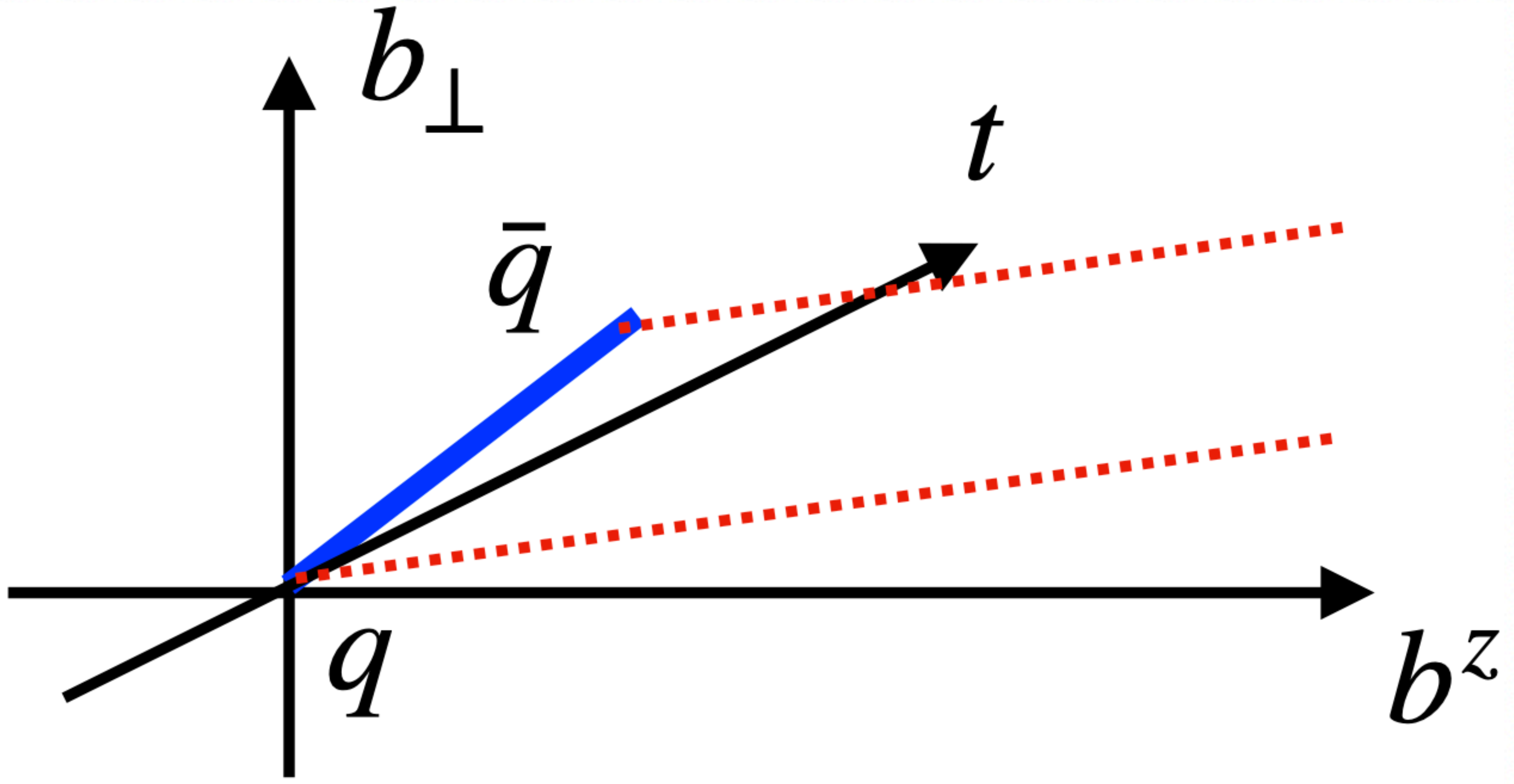
$$\lim_{\alpha\to 1} \mathscr{C}^{(1)}(\alpha, z^2\mu^2)$$
$$\sim \frac{\alpha_s C_F}{2\pi}\left[\underbrace{-\frac{4\ln(1-\alpha)}{1-\alpha} - \frac{2}{(1-\alpha)}\ln\frac{z^2\mu^2 e^{2\gamma_E}}{4}}_{\frac{2}{1-\alpha}\ln\frac{4e^{-2\gamma_E}}{(1-\alpha)^2 z^2\mu^2}} - \frac{2}{1-\alpha}\right]_+$$

$\alpha \to 1$ approaching UV fixed point

Origin of threshold logarithms

3D momentum distribution

$$\tilde{q}(x, \vec{k}_\perp, P^z) = \frac{1}{2P^0} \int \frac{db_z d^2\vec{b}_\perp}{(2\pi)^3} e^{i\vec{k}_\perp \cdot \vec{b}_\perp + ib_z(xP^z)} \times \langle P | \bar{\psi}(b) W(b,0) \gamma^t \psi(0) | P \rangle$$



- Straight-line gauge link
- Different from normal TMD distribution with a staple shaped gauge link

Origin of threshold logarithms

3D momentum distribution

$$\tilde{q}(x, \vec{k}_\perp, P^z) = \frac{1}{2P^0}\int \frac{db_z d^2\vec{b}_\perp}{(2\pi)^3} e^{i\vec{k}_\perp\cdot\vec{b}_\perp + ib_z(xP^z)}$$
$$\times \langle P|\bar{\psi}(b)W(b,0)\gamma^t\psi(0)|P\rangle$$

- **Relation to the quasi-PDF**

$$\int d^2\vec{k}_\perp\ \tilde{q}(x, \vec{k}_\perp, P^z) = \tilde{q}(x, P^z)$$

or

$$\lim_{b_\perp \to 0} \tilde{q}(x, \vec{b}_\perp, P^z) = \tilde{q}(x, P^z)$$

- **Relation to the pseudo-PDF**

$$\frac{1}{2P^0}\langle P|\bar{\psi}(b)W(b,0)\gamma^t\psi(0)|P\rangle = h(P\cdot b, b^2)$$

spacial correlator

In the limit of $P_z \to \infty,\ b_z \to 0,\ \lambda = P_z b_z$ finite

$$\lim_{P^z\to\infty} \tilde{q}(x, \vec{b}_\perp, P^z)$$
$$= \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\ e^{i\lambda x}\tilde{h}_{\gamma^t}(\lambda, -b^2 = \vec{b}_\perp^2)$$
$$= \mathscr{P}(x, \vec{b}_\perp^2)$$

primordial TMD

- A. V. Radyushkin, Phys.Rev.D 96 (2017)

$$\vec{b}_\perp^2 = z^2$$

Origin of threshold logarithms

- **Quasi-PDF**

$$\frac{2}{|1-\xi|}\ln\frac{|1-\xi|P_z^2}{\mu^2}$$

$\xi \to 1$ approaching Landau pole

Since $k_\perp$ is integrated over, the limit $x \to 1$ includes contributions from both hard and soft transverse momentum modes, with the latter being sensitive to IR physics.

- **Pseudo-PDF**

$$\frac{2}{1-\alpha}\ln\frac{4e^{-2\gamma_E}}{(1-\alpha)^2 z^2 \mu^2}$$

$\alpha \to 1$ approaching UV fixed point

Since pPDF corresponds to the primordial TMD, the emitted gluon remains off-shell with virtual mass $k_\perp$ in the limit of $x \to 1$. In coordinate space, small $b_\perp$ corresponds to large $k_\perp$, so the gluon is in the UV region.

$$\int d^2\vec{k}_\perp\, \tilde{q}(x, \vec{k}_\perp, P^z) = \tilde{q}(x, P^z)$$

$$\lim_{P^z\to\infty} \tilde{q}(x, \vec{b}_\perp, P^z)$$

Threshold resummation at NLL accuracy

In the Mellin-moment space (OPE)

$$a_N(\mu) = \int_{-1}^{1} dy\ y^N q(y,\mu)$$

$$\tilde{h}_{\gamma^t}(\lambda, z^2\mu^2) = \sum_{N=0}^{\infty} \frac{(-i\lambda)^N}{N!} C_N(\alpha_s(\mu), z_0^2\mu^2) a_N(\mu) + \ldots$$

- **At NLO**

$$C_N^{\rm NLO} = \int_0^1 dw\ w^N \mathscr{C}^{\rm NLO}(w, z^2\mu^2)$$

$$= \frac{\alpha_s(\mu)C_F}{2\pi}\left[\left(\frac{3+2N}{2+3N+N^2}+2H_N\right)\ln(z_0^2\mu^2)\right.$$

$$\left. + \frac{5+2N}{2+3N+N^2} + 2(1-H_N)H_N - 2H_N^{(2)}\right]$$

- **The threshold limit corresponds to $N \to \infty$**

$$\lim_{N\to\infty} C_N^{\rm NLO} = \frac{\alpha_s(\mu)C_F}{2\pi}\left[2\ln N' \ln(z_0^2\mu^2)\right.$$

$$\left. -2\ln^2 N' + 2\ln N' - \frac{\pi^2}{3}\right],$$

$\underset{\rm LL}{\alpha_s \ln^2 N'}$, $\underset{\rm NLL}{\alpha_s \ln N'}$ $\qquad N' = Ne^{\gamma_E}$

Threshold resummation at NLL accuracy

Using the standard technique of threshold resummation

$$\ln C_N^{\mathrm{NLL}} = \int dx \frac{x^{N-1}-1}{1-x}\left[\int_{\mu^2}^{\frac{(1-x)^{-2}}{z_0^2}} \frac{dk^2}{k^2} A(\alpha_s(k^2)) + B(\alpha_s((1-x)^{-2}/z_0^2))\right]$$

$$A(\alpha_s) = A^{(0)}a_s + A^{(1)}a_s^2 + \dots\,, \quad B(\alpha_s) = B^{(0)}a_s + B^{(1)}a_s^2 + \dots\,.$$

- **Leading logarithm (LL)**

$$A^{(0)} = -B^{(0)} = 2C_F$$

- **For NLL which neglects $\mathcal{O}(\alpha_s^2 \ln N')$ terms**

$$A^{(1)} = 2C_F\left[C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{10}{9}n_f T_F\right]$$

- **DGLAP evolution may also be considered**

$$\left[\frac{\partial}{\partial \ln \mu^2} + \beta(a_s(\mu))\frac{\partial}{\partial a_s} - \gamma_N\right] C_N = 0$$

Threshold resummation at NLL accuracy

- NLL threshold resummation + LL DGLAP evolution (evo)

$$C_N^{\mathrm{NLL+evo}}(\alpha_s(\mu), z_0^2\mu^2) = C_N^{\mathrm{NLL}}\left(\alpha_s(z_0^{-1}),1\right)\left(\frac{\alpha_s(z_0^{-1})}{\alpha_s(\mu)}\right)^{\frac{\gamma_N^{(0)}}{\beta_0}}$$

$$\ln C_N^{\mathrm{NLL}}\left(\alpha_s(z_0^{-1}),1\right) = -\frac{\pi^2}{3}a_s C_F + \ln N' g_1(\tau,0) + g_2(\tau,0)$$

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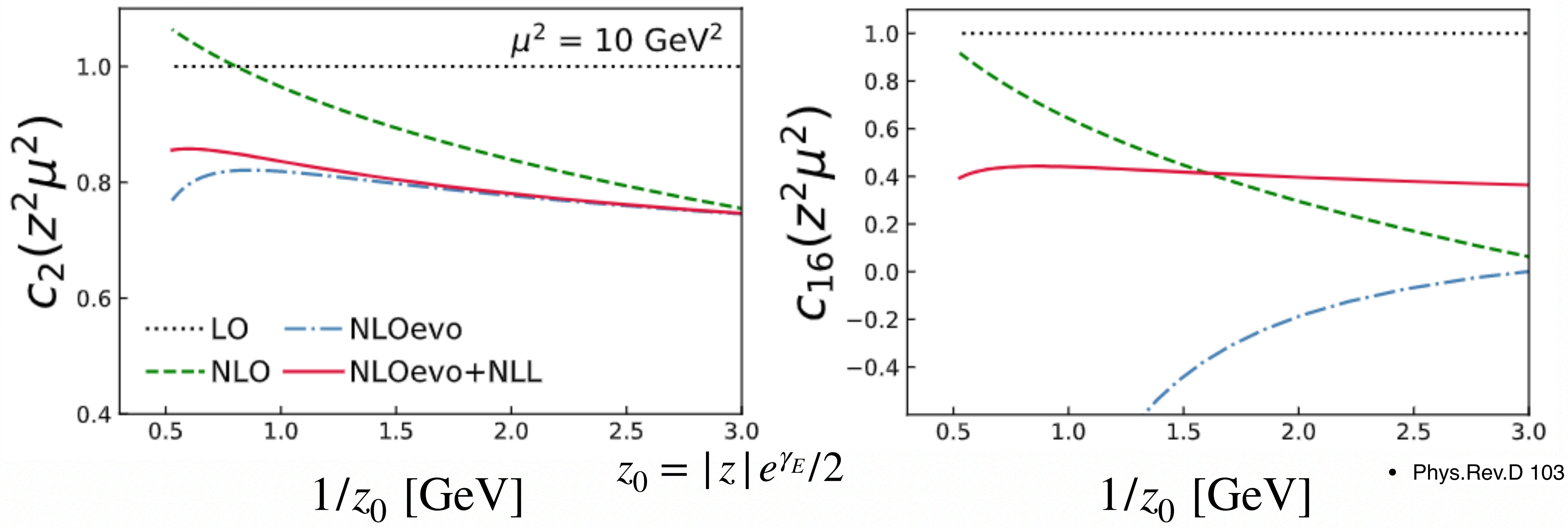
- Using the inverse Mellin transform, we can eventually obtain the resummed matching coefficient

$$\mathscr{C}^{\mathrm{NLL+evo}}(w, z^2\mu^2) = e^{-\frac{\pi^2}{3}a_s C_F}\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dN\, w^{-N}$$

$$\times \exp\left[\ln N' g_1(\tau,0) + g_2(\tau,0)\right]\left(\frac{\alpha_s(z_0^{-1})}{\alpha_s(\mu)}\right)^{\frac{\gamma_N^{(0)}}{\beta_0}}$$

Impacts of NLL resummation

The Wilson coefficient at LO, NLO, NLOevo, and NLOevo+NLL accuracy



$z_0 = |z|e^{\gamma_E}/2$

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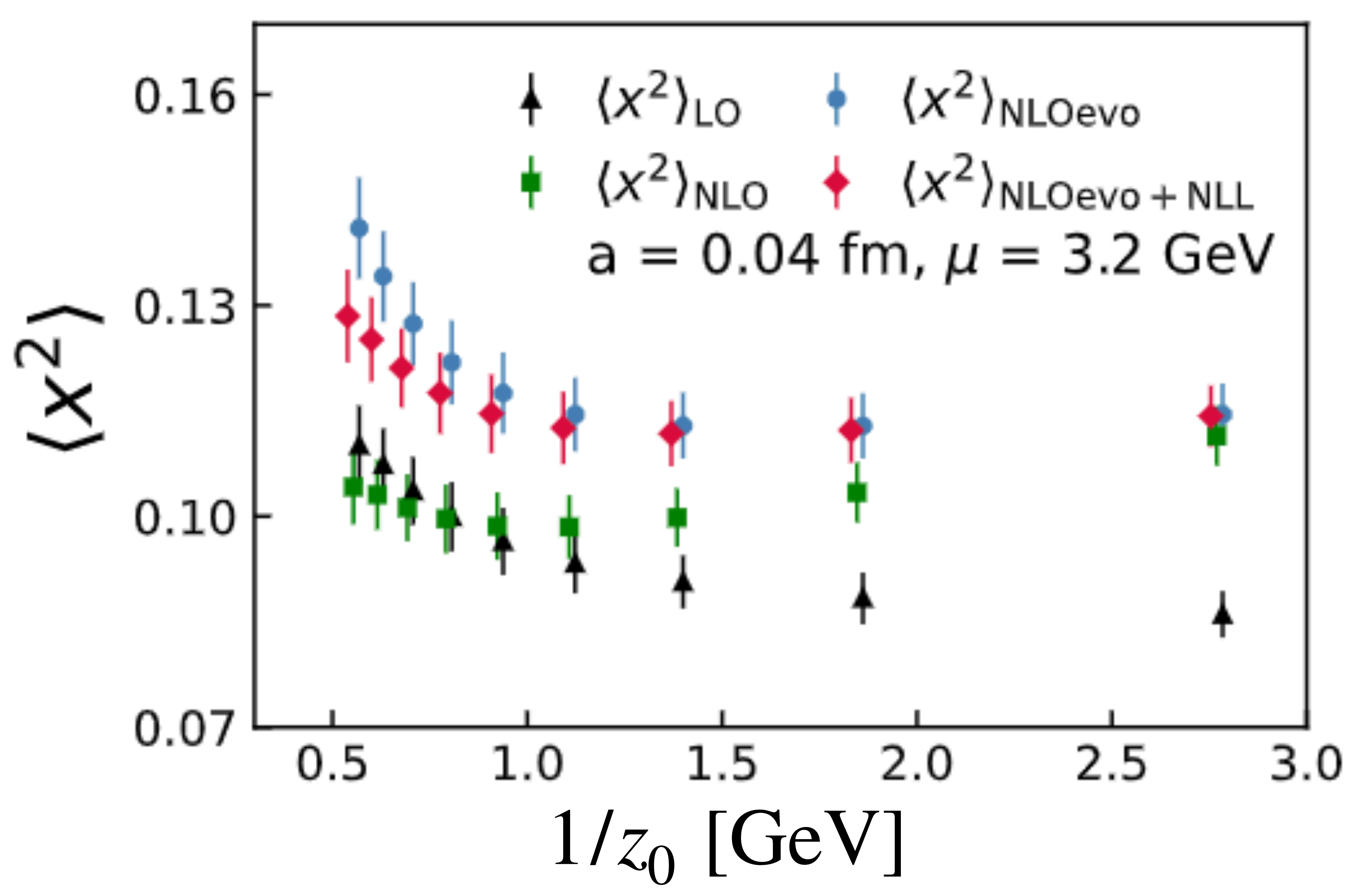
- DGLAP evolution is important when $1/z_0$ is far from μ.
- Threshold resummation is necessary for either large α_s or large N

$\alpha_s \ln^2 N'$, $\alpha_s \ln N'$

Impacts of NLL resummation

$\langle x^2 \rangle$ **by fitting** $a = 0.04$ fm **lattice results with pion boosted up to 2.42 GeV**

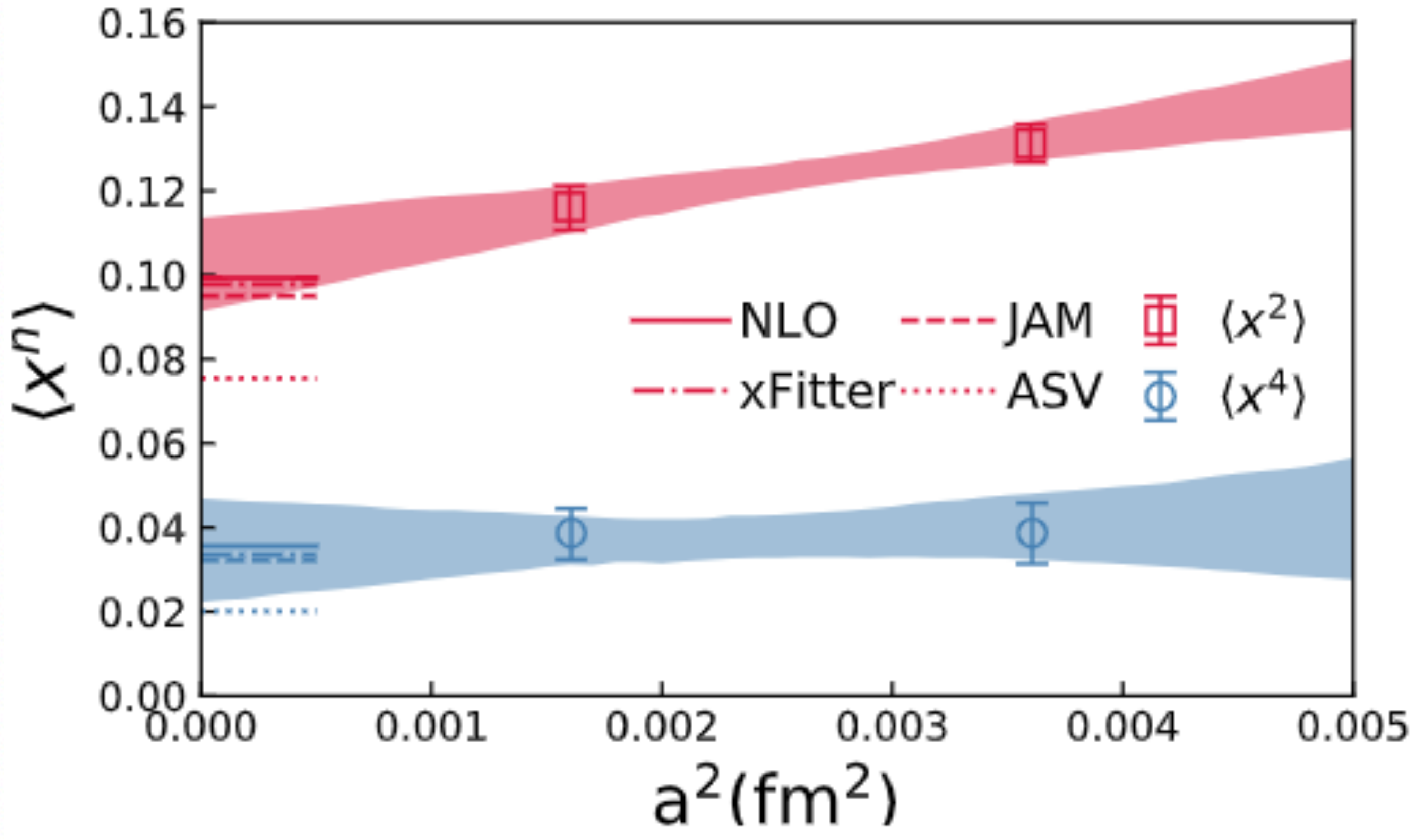


$z_0 = |z| e^{\gamma_E}/2$

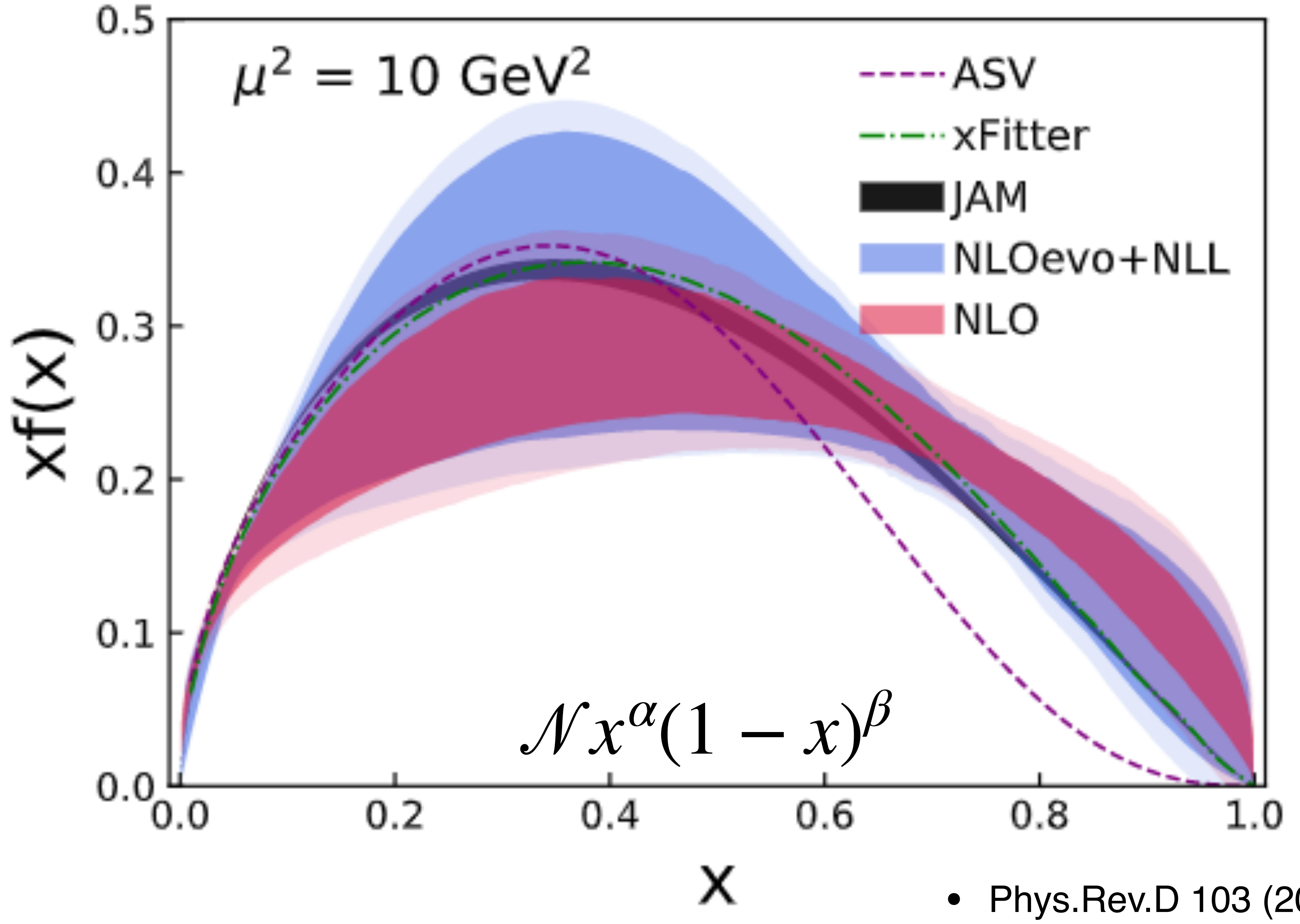
- At LO one can clearly observe the z-dependence.
- Beyond LO, one can find a plateau indicating that the coefficients can explain the z-dependence.
- Threshold resummation slightly improve the plateau.

Impacts of NLL resummation

Moments of pion valence quark PDF



Pion valence quark PDF



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- Our current lattice data are only sensitive to the first few moments, where threshold resummation has mild impact.
- Situation can be improved if we manage to get more precise data and increase the pion momentum aimed for higher moments.

Summary

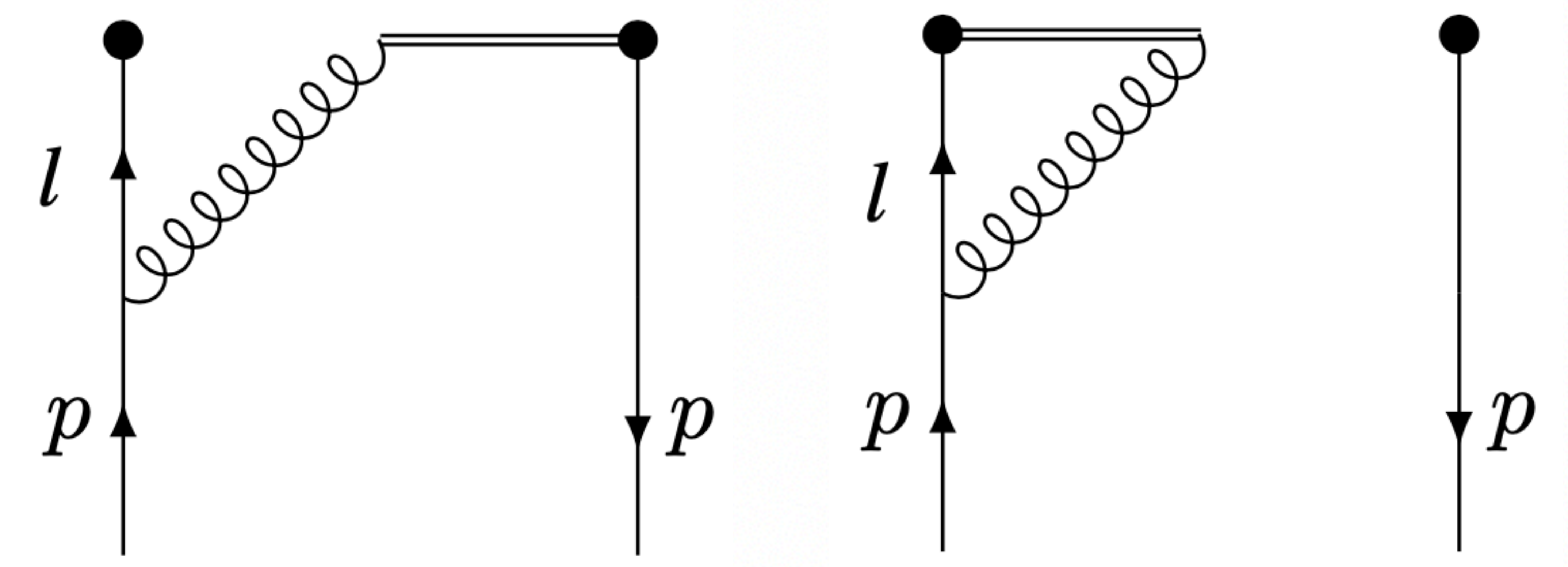
- Precision lattice calculation of PDFs will require QCD evolution and resummation.
- The origin of threshold logarithms in the quasi-PDF and spatial correlators is identified, and resummed using standard techniques.
- Current lattice data are only sensitive to the lowest moments or finite-x range of the PDF, so the effect of threshold resummation is not significant.
- Threshold resummation will be important for future calculations with larger hadron momenta to study the large-x behavior of the PDF.

Origin of threshold logarithms

Leading divergence in the one-loop diagram

$$\tilde{q}^{(1)}_{\rm cs}(x, \vec{k}_\perp, p^z)$$
$$= \frac{g^2 \mu^{2\epsilon} C_F}{2(2\pi)^{d-1}} \int_0^1 ds \; \frac{(1-s)^{2-d}}{\vec{k}_\perp^2}$$
$$\times \left[\frac{k_t^2(1+x-2s)+(x-s)^3}{\left(k_t^2+(s-x)^2\right)^{3/2}} - \frac{k_t^2(1+x-2s)+(x-1)^3}{\left(k_t^2+(x-1)^2\right)^{3/2}} \right]$$

$$k_t^2 = \vec{k}_\perp^2 / p_z^2$$



- **To obtain the quasi-PDF**

$$\tilde{q}^{(1)}_{\rm cs}(x, p^z) = \int d^{d-2}k_\perp \; \tilde{q}^{(1)}_{\rm cs}(x, \vec{k}_\perp, p^z)$$
$$\xRightarrow{x \to 1^-} -\frac{g^2 C_F}{8\pi^2} \frac{1}{\epsilon} \frac{1+(1-x)^{-2\epsilon}}{1-x} \left(\frac{\mu^2}{p_z^2}\right)^{\epsilon}$$

Here the factor $(1-x)^{-2\epsilon}$ is crucial to reproduce the correct sign of leading threshold logarithm, which plays a similar role as the phase-space measure in DIS and DY cross sections.

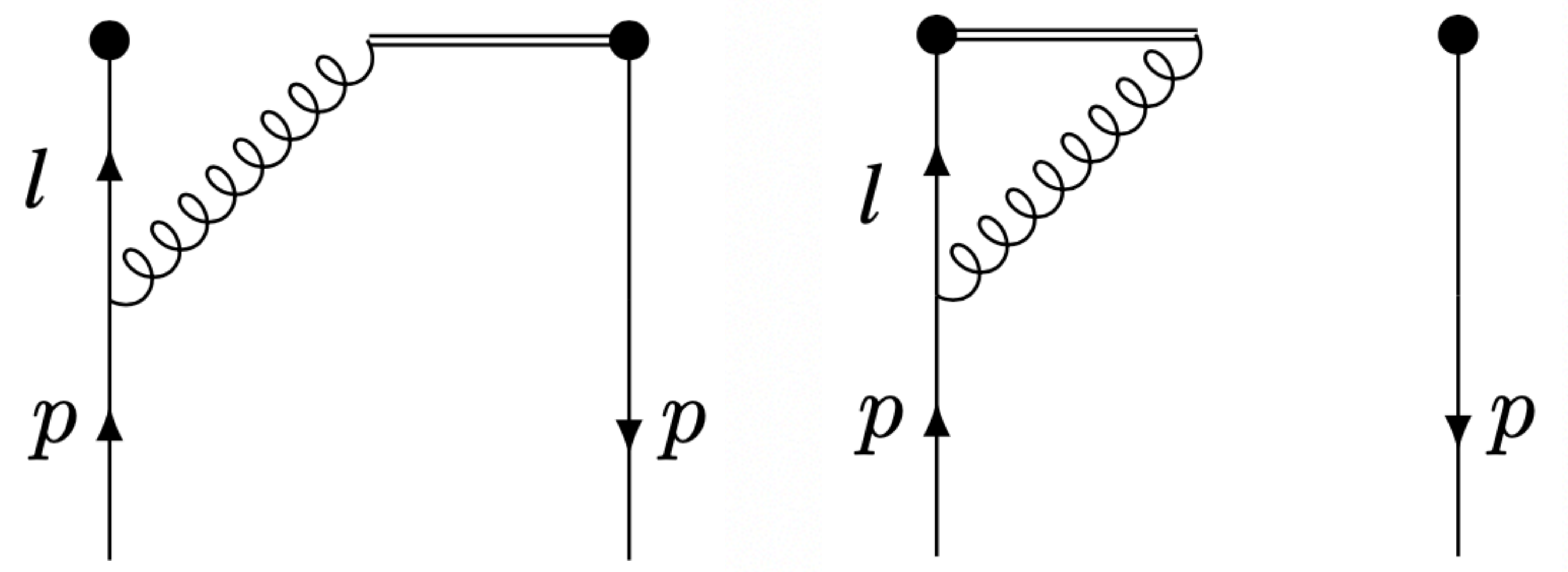
By expanding in ϵ, we can reproduce the leading threshold logarithm.

Origin of threshold logarithms

Leading divergence in the one-loop diagram

$$\tilde{q}_{\rm cs}^{(1)}(x,\vec{k}_\perp,p^z)$$
$$=\frac{g^2\mu^{2\epsilon}C_F}{2(2\pi)^{d-1}}\int_0^1 ds\,\frac{(1-s)^{2-d}}{\vec{k}_\perp^2}$$
$$\times\left[\frac{k_t^2(1+x-2s)+(x-s)^3}{\left(k_t^2+(s-x)^2\right)^{3/2}}-\frac{k_t^2(1+x-2s)+(x-1)^3}{\left(k_t^2+(x-1)^2\right)^{3/2}}\right]$$

$$k_t^2=\vec{k}_\perp^2/p_z^2$$



- **To obtain the pseudo-PDF**

$$\lim_{x\to 1}\tilde{q}_{\rm cs}^{(1)}(x,\vec{b}_\perp,p^z=\infty)$$
$$=\frac{g^2C_F}{8\pi^2}\Gamma(-\epsilon)\frac{2(1-x)^{2\epsilon}}{(1-x)}(b_\perp^2\mu^2)^\epsilon$$

Since the factorization for the pPDF in the small $b_\perp$ limit, the physical scale in the threshold logarithm is proportional to $(1-x)^{-2}b_\perp^{-2}$, which approaches the UV fixed point in the $x\to 1$ limit.

By expanding in ϵ, we can reproduce the leading threshold logarithm.