Resuming Small-Momentum Large Logarithms in LaMET PDF Matching

Yushan Su, University of Maryland

Y. Su, J. Holligan, X. Ji, F. Yao, J. Zhang, R. Zhang, arXiv: 2209.01236

High Precision LaMET PDFs from Lattice

High Precision

Lattice Data

- Large Momentum Limit
- Excited State Contamination
- Discretization Effect (O(a) Effect, O(a/z))
- Physical Pion Mass
- Infinite Volume Limit

High Accuracy

Perturbation Theory

- Hybrid Renormalization and Self Renormalization
- Renormalon Uncertainty and Power Accuracy
- Small-Momentum Logarithms Resummation($\Lambda_{\mathrm{QCD}} \ll p_z \ll P_z$)
- Threshold Logarithms Resummation ($p_z \sim P_z$)

Outline

- Motivations for Resummation
- Resummation in LaMET PDF Matching
- Resummation Effects in Pion PDF

QCD Resummation for a single physics scale

- QCD perturbation series with single log terms

(LaMET PDF matching kernel is under this framework)

$$
\begin{aligned}
\tilde{h}(Q,\mu) = 1 &+ \alpha_s(\mu)[c_{11}L + c_{10}] \\
&+ \alpha_s^2(\mu)[c_{22}L^2 + c_{21}L + c_{20}] \\
&+ \alpha_s^3(\mu)[c_{33}L^3 + c_{32}L^2 + c_{31}L + c_{30}] \\
&+ \alpha_s^4(\mu)[c_{44}L^4 + c_{43}L^3 + c_{42}L^2 + c_{41}L + c_{40}] \\
&+ \cdots
\end{aligned}
$$

Q: physics scale

μ: renormalization scale

$L = \ln(\mu^2/Q^2)$

If Q differs from μ by orders of magnitude, $\alpha_s(\mu)\, L$ is large.

e.g. $Q = 10\ \text{GeV}$ and $\mu = 2\ \text{GeV}$, $\alpha_s(\mu)\, L \sim O(1)$.

One has to resum the log terms.

QCD Resummation

- QCD perturbation series with single log terms

$$\tilde{h}(Q,\mu) = 1 + \alpha_s(\mu)[c_{11}L + c_{10}] + \alpha_s^2(\mu)[c_{22}L^2 + c_{21}L + c_{20}] + \cdots$$

Q: physics scale
μ: renormalization scale
$L = \ln(\mu^2/Q^2)$

- Renormalization group equation constrains the coefficients of log terms

$$\frac{d\tilde{h}(Q,\mu)}{d\ln\mu^2} = \gamma(\mu)\tilde{h}(Q,\mu)$$

⇓

Recursive relations between the coefficients

⇓

Resummation of the log terms

QCD Resummation

- QCD perturbation series with single log terms

$$\tilde{h}(Q,\mu) = 1 + \alpha_s(\mu)[c_{11}L + c_{10}] + \alpha_s^2(\mu)[c_{22}L^2 + c_{21}L + c_{20}] + \cdots$$

Q: physics scale
μ: renormalization scale
$L = \ln(\mu^2/Q^2)$

- The log terms are generated from RG evolution

$$\tilde{h}_{\mathrm{RGR}}(Q,\mu) = \tilde{h}(Q,Q)\exp\left[\int_Q^\mu d\ln(\mu'^2)\ \gamma(\mu')\right]$$

Solving the RG equation is equivalent to resuming the log terms

If $\mu \sim Q$, use the infinitesimal transformation

$$\exp\left[\int_Q^\mu d\ln(\mu'^2)\ \gamma(\mu')\right] \sim 1 + \int_Q^\mu d\ln(\mu'^2)\ \gamma(\mu')$$

If $\mu \gg Q$ or $\mu \ll Q$, keep $\exp\left[\int_Q^\mu d\ln(\mu'^2)\ \gamma(\mu')\right]$

e.g. Leading Log Resummation

- QCD perturbation series with large log terms

$$\tilde{h}(Q,\mu) = 1 + \alpha_s(\mu)[c_{11}L + c_{10}] + \alpha_s^2(\mu)[c_{22}L^2 + c_{21}L + c_{20}] + \alpha_s^3(\mu)[c_{33}L^3 + c_{32}L^2 + c_{31}L + c_{30}] + \cdots$$

LO+RGR

$\gamma_0,\ b_0$

$Q:$ physics scale
$\mu:$ renormalization scale
$L = \ln(\mu^2/Q^2)$

- Leading log resummation through recursive relation

$$\frac{d\tilde{h}(Q,\mu)}{d\ln\mu^2} = \gamma(\mu)\tilde{h}(Q,\mu)$$

$$\frac{d\alpha_s}{d\ln\mu^2} = -b_0\alpha_s^2 + \cdots$$

$$\gamma = \gamma_0\alpha_s + \cdots$$

$$\Rightarrow \quad c_{m+1,m+1} = c_{m,m}\frac{\gamma_0 + b_0 m}{m+1}$$

$$\tilde{h}_{\mathrm{LO+RGR}}(Q,\mu) = 1 + \alpha_s(\mu)\gamma_0 L + \alpha_s^2(\mu)\frac{1}{2}\gamma_0(b_0+\gamma_0)L^2 + \cdots = \left(\frac{1}{1 - b_0\,\alpha_s(\mu)\,L}\right)^{\gamma_0/b_0}$$

- Leading log resummation through RG evolution

physics scale appears in the running coupling

$$\tilde{h}_{\mathrm{LO+RGR}}(Q,\mu) = \tilde{h}(Q,Q)\exp\left[\int_Q^\mu d\ln(\mu'^2)\ \gamma(\mu')\right] = \left(\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right)^{\gamma_0/b_0} = \left(\frac{1}{1 - b_0\,\alpha_s(\mu)\,L}\right)^{\gamma_0/b_0}$$

e.g. Leading Log Resummation

- QCD perturbation series with large log terms

$$\tilde{h}(Q,\mu) = 1 + \alpha_s(\mu)[c_{11}L + c_{10}] + \alpha_s^2(\mu)[c_{22}L^2 + c_{21}L + c_{20}] + \alpha_s^3(\mu)[c_{33}L^3 + c_{32}L^2 + c_{31}L + c_{30}] + \cdots$$

Q: physics scale
μ: renormalization scale
$L = \ln(\mu^2/Q^2)$

LO+RGR: γ_0, b_0
NLO+RGR: $\gamma_{0,1}, b_{0,1}, c_{10}$
NNLO+RGR: $\gamma_{0,1,2}, b_{0,1,2}, c_{10}, c_{20}$

- Leading log resummation through recursive relation

$$\frac{d\tilde{h}(Q,\mu)}{d\ln\mu^2} = \gamma(\mu)\tilde{h}(Q,\mu)$$
$$\frac{d\alpha_s}{d\ln\mu^2} = -b_0\alpha_s^2 + \cdots$$
$$\gamma = \gamma_0\alpha_s + \cdots$$

$$\Rightarrow \quad c_{m+1,m+1} = c_{m,m}\frac{\gamma_0 + b_0 m}{m+1}$$

$$\tilde{h}_{\mathrm{LO+RGR}}(Q,\mu) = 1 + \alpha_s(\mu)\gamma_0 L + \alpha_s^2(\mu)\frac{1}{2}\gamma_0(b_0+\gamma_0)L^2 + \cdots = \left(\frac{1}{1-b_0\,\alpha_s(\mu)\,L}\right)^{\gamma_0/b_0}$$

- Leading log resummation through RG evolution

physics scale appears in the running coupling

$$\tilde{h}_{\mathrm{LO+RGR}}(Q,\mu) = \tilde{h}(Q,Q)\exp\left[\int_Q^\mu d\ln(\mu'^2)\ \gamma(\mu')\right] = \left(\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right)^{\gamma_0/b_0} = \left(\frac{1}{1-b_0\,\alpha_s(\mu)\,L}\right)^{\gamma_0/b_0}$$

Log terms in $\overline{\text{MS}}$ scheme matching kernel

- LaMET factorization

$$\tilde{f} = C \otimes f$$

where $\tilde{f}$ is the quasi-PDF, f is the light-cone PDF and C is the matching kernel.

- e.g. Leading Log terms in the NLO, NNLO matching kernels

X. Xiong et al., PRD (2014)
T. Izubuchi et al., PRD (2018)
W. Wang et al., PRD (2019)
Z. Li et al., PRL (2021)
L. Chen et al., PRL (2021)

$$C^{\overline{\text{MS}}}_{\text{NLO}}\left(\xi, \frac{\mu}{|x|P_z}\right) = \frac{\alpha_s(\mu)C_F}{2\pi}\frac{1+\xi^2}{1-\xi}(-L)\theta(1-\xi)\theta(\xi) + \frac{\alpha_s(\mu)C_F}{2\pi}\delta(1-\xi)\left(\frac{3}{2}L + \frac{5}{2}\right) + \cdots$$

$$C^{\overline{\text{MS}}}_{\text{NNLO}}\left(\xi, \frac{\mu}{|x|P_z}\right) = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 C_F^2 \frac{(3\xi^2+1)\ln[\xi]-4(1+\xi^2)\text{Log}[1-\xi]+2(1-\xi)^2}{2(\xi-1)} L^2\theta(1-\xi)\theta(\xi)$$

$$+\left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \frac{1+\xi^2}{2(\xi-1)} C_F b_0 2\pi L^2 \theta(1-\xi)\theta(\xi) + \cdots$$

$\xi = \frac{x}{y}$

$2xP_z$: physics scale

μ: renormalization scale

$L = \ln\left(\frac{\mu^2}{4x^2P_z^2}\right)$

- Leading Log terms satisfy the recursive relations

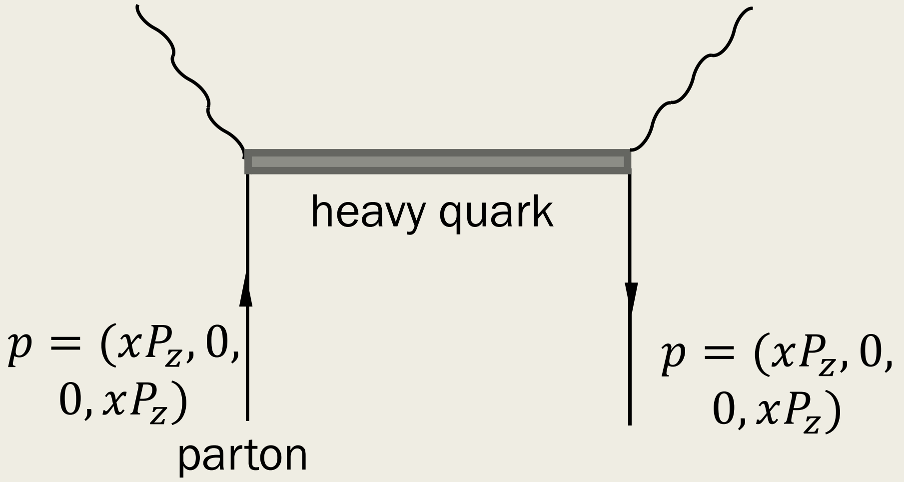
$$\left.\begin{aligned} \tilde{f} &= C \otimes f \\ \frac{d\tilde{f}}{d\ln\mu^2} &= \gamma\tilde{f} \\ \frac{df}{d\ln\mu^2} &= P \otimes f \end{aligned}\right\} \Rightarrow \frac{dC}{d\ln\mu^2} = -C\otimes P + \gamma C \Rightarrow c_{m+1,m+1} = c_{m,m}\frac{-\otimes P_0 + \gamma_0 + b_0 m}{m+1}$$

DGLAP kernel (P)

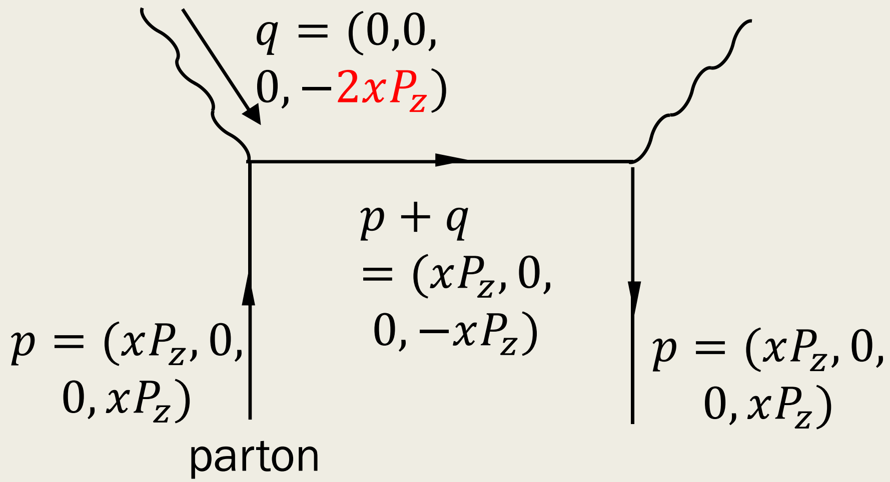
Heavy light quark anomalous dimension (γ)

Why the physics scale is $2xP_z$?

LaMET



DIS



$$|P_z = \infty\rangle = \Lambda(\infty)\,|P_z = \infty\rangle$$

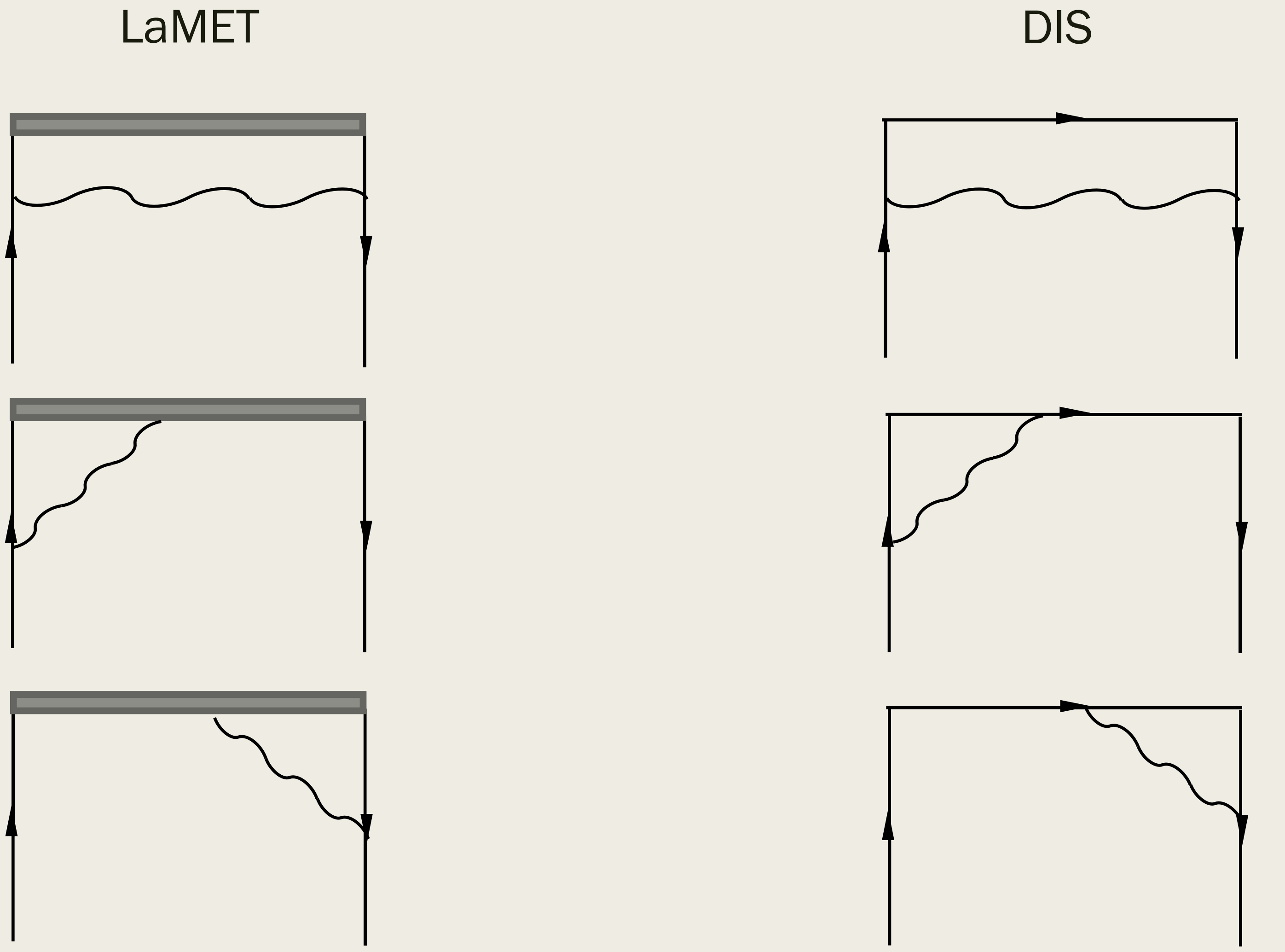
$$\langle P_z = \infty|\bar{\psi}(z)U(z,0)\gamma^t\psi(0)|P_z = \infty\rangle \Longrightarrow \langle P_z = \infty|\bar{\psi}(n\lambda)U(n\lambda,0)\gamma^+\psi(0)|P_z = \infty\rangle$$

- In Bjorken frame:

$$q^\mu = (0,0,0,-2xP_z)$$
$$p^\mu = (xP_z, 0,0, xP_z)$$
$$(p+q)^\mu = (xP_z, 0,0, -xP_z)$$

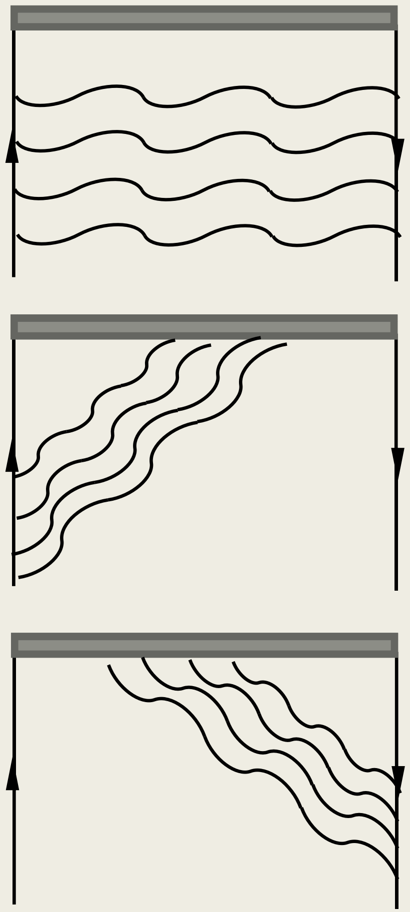
- The virtuality of photon $Q^2 = (2xP_z)^2$

Leading Logs in the Feynman Diagrams

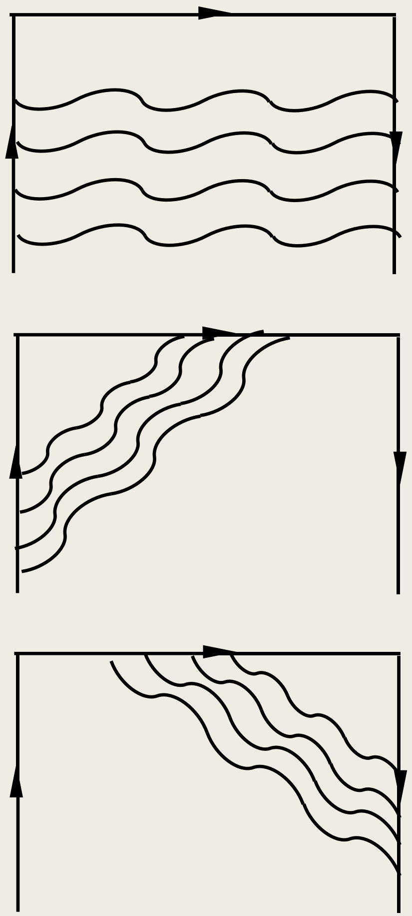


Leading Logs in the Feynman Diagrams

LaMET



DIS



Resummation in Hybrid scheme

- Hybrid renormalized matrix element is scale independent: X. Ji et al., NPB (2021)

$$\tilde{h}(z,P_z) = \frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{\tilde{h}^{\text{lat}}(z,a,0)}\theta(z_s-|z|) + \frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{Z^R(z,a,\mu)\tilde{h}^{\overline{\text{MS}}}(z_s,\mu,0)}\theta(|z|-z_s)$$

- RG equation

$$f = C^{-1}\otimes\tilde{f}$$

$$\frac{d\tilde{f}}{d\ln\mu^2} = 0$$

$$\frac{df}{d\ln\mu^2} = P\otimes f$$

$$\Rightarrow \quad \frac{dC^{-1}}{d\ln\mu^2} = P\otimes C^{-1}$$

DGLAP kernel

- RG equation

$$\mu\frac{dC^{-1}_{\text{RGR}}\left(\frac{x}{y},\frac{\mu}{|x|P_z}\right)}{d\mu} = \int_x^1 \frac{dw}{w}P[w,\alpha_s(\mu)]C^{-1}_{\text{RGR}}\left(\frac{x/w}{y},\frac{\mu}{|x/w|P_z}\right)$$

where the initial condition is

$$C^{-1}_{\text{RGR}}\left(\frac{x}{y},\frac{2xP_z}{|x|P_z}\right) = C^{-1}_{\text{fixed order}}\left(\frac{x}{y},\frac{2xP_z}{|x|P_z}\right)$$

physics scale appears in the running coupling $\alpha_s(2xP_z)$

Lattice Data and Renormalization

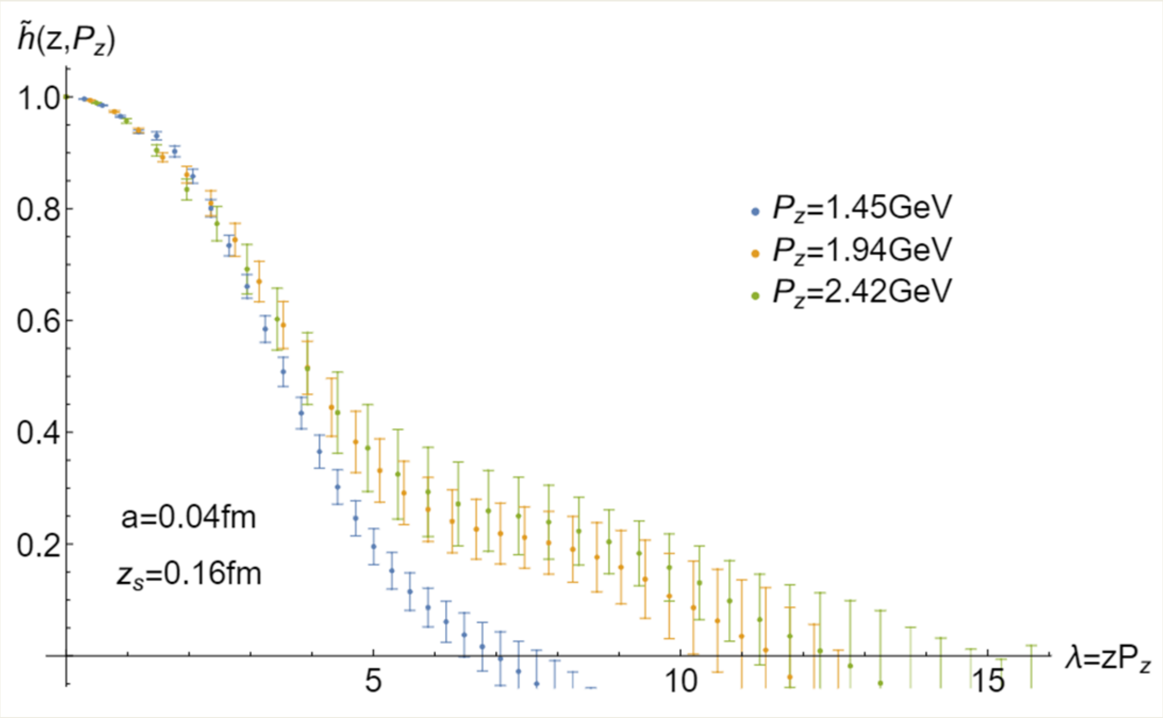
- Pion valence PDF matrix element from BNL/ANL collaboration:

$$\tilde{h}^{\text{lat}}(z,a,P_z) = \langle \pi^+(P_z)|\bar{u}(z)\gamma^t U(z,0)u(0) - \bar{d}(z)\gamma^t U(z,0)d(0)|\pi^+(P_z)\rangle$$

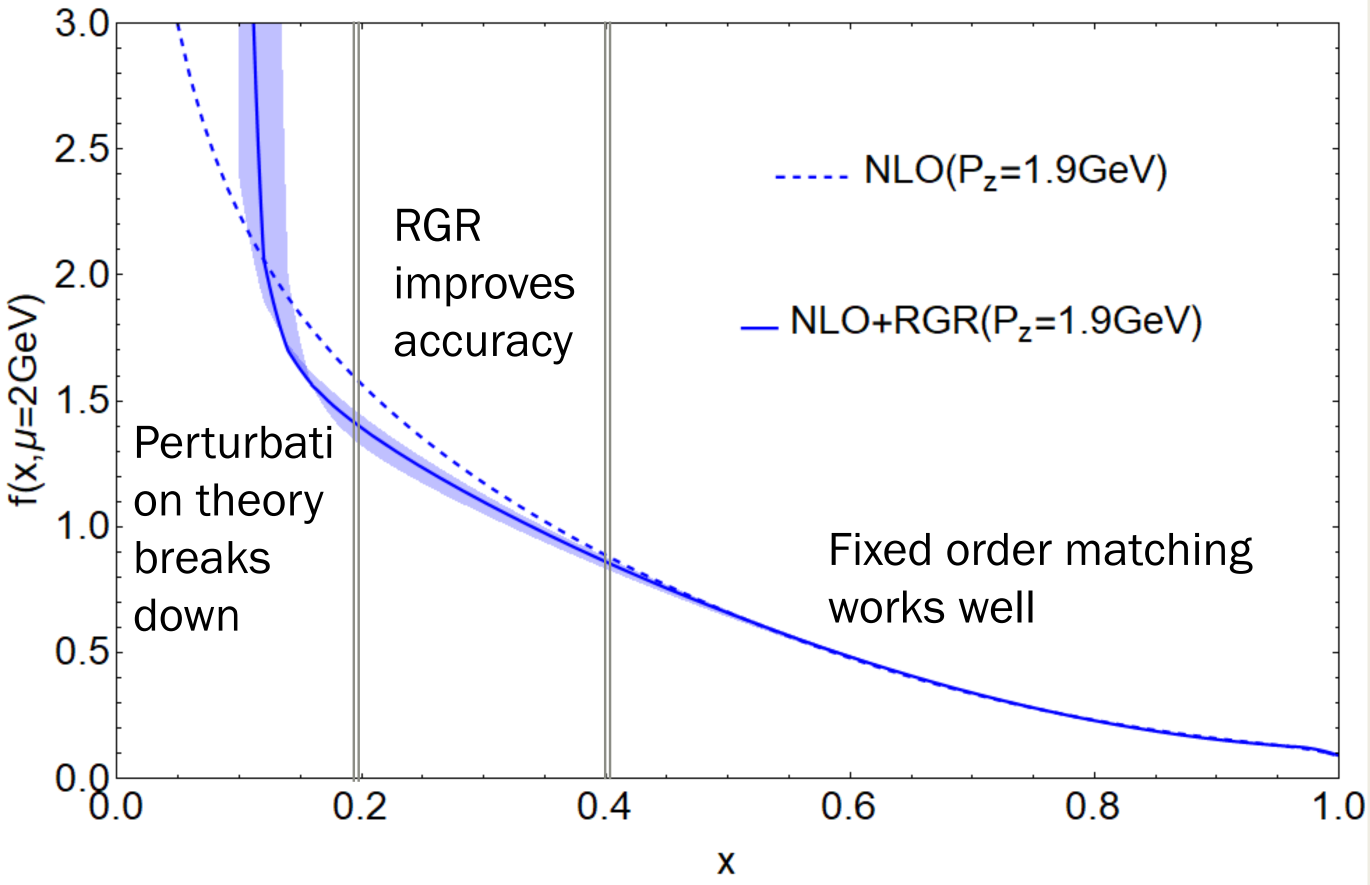
- Hybrid renormalized matrix element:

$$\tilde{h}(z,P_z) = \frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{\tilde{h}^{\text{lat}}(z,a,0)}\theta(z_s-|z|) + \frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{Z^R(z,a,\mu)\tilde{h}^{\overline{\text{MS}}}(z_s,\mu,0)}\theta(|z|-z_s)$$

Gao et al., PRD (2020)
Gao et al., PRD (2021)
Gao et al., PRL (2022)



Compare RGR matching with fixed order one



NLO

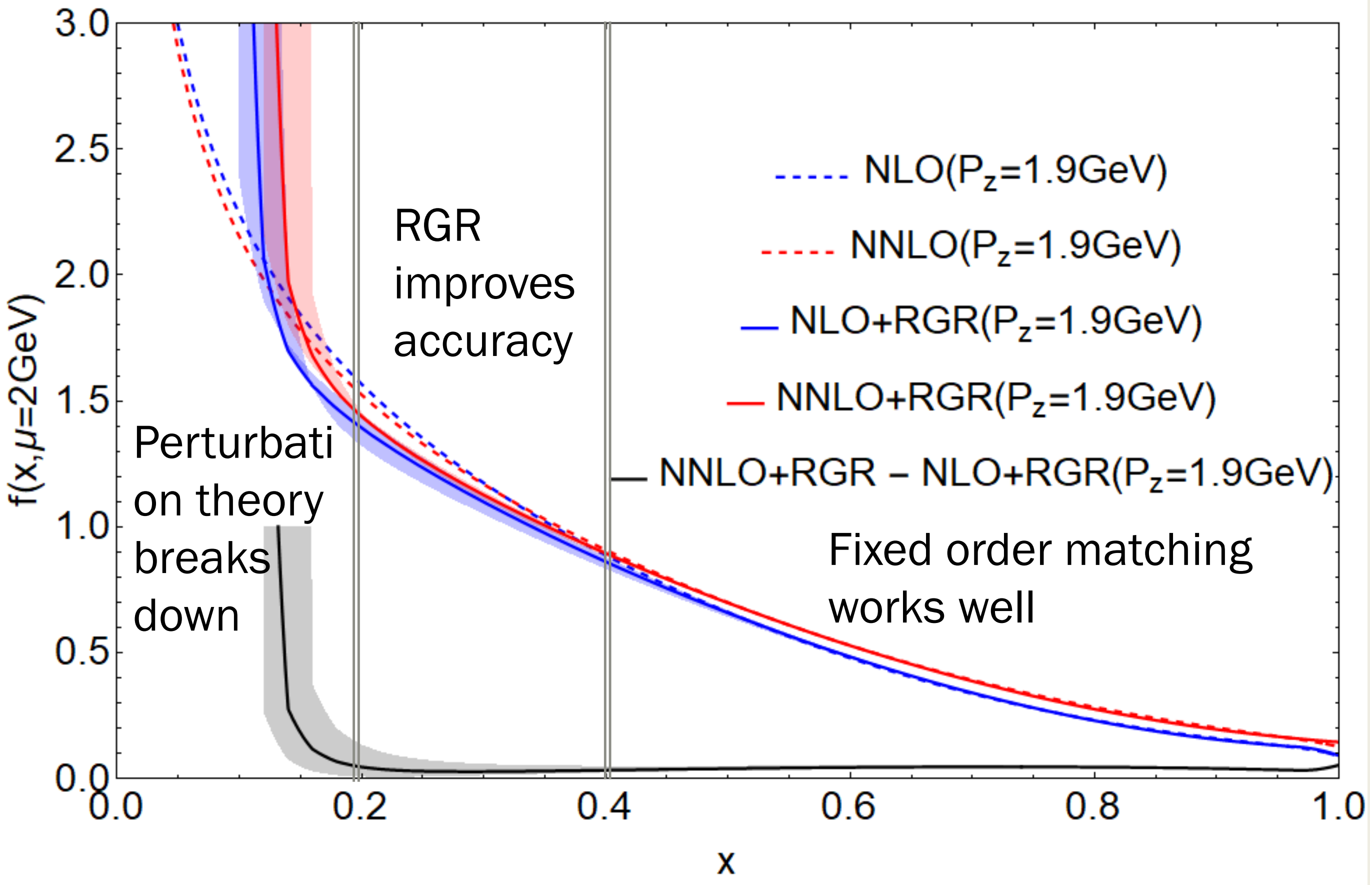
$$1 + \alpha_s(\mu)[c_{11}L + c_{10}] + \alpha_s^2(\mu)[c_{22}L^2 + c_{21}L + c_{20}] + \alpha_s^3(\mu)[c_{33}L^3 + c_{32}L^2 + c_{31}L + c_{30}] + \cdots$$

NLO+RGR

$$C_{\mathrm{RGR}}^{-1}\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right) = e^{P\cdots}\left[\delta\left(1-\frac{x}{y}\right) + \alpha_s(2xP_z c') * \cdots\right]$$

$c' = 0.8 \sim 1.2$

Compare RGR matching with fixed order one



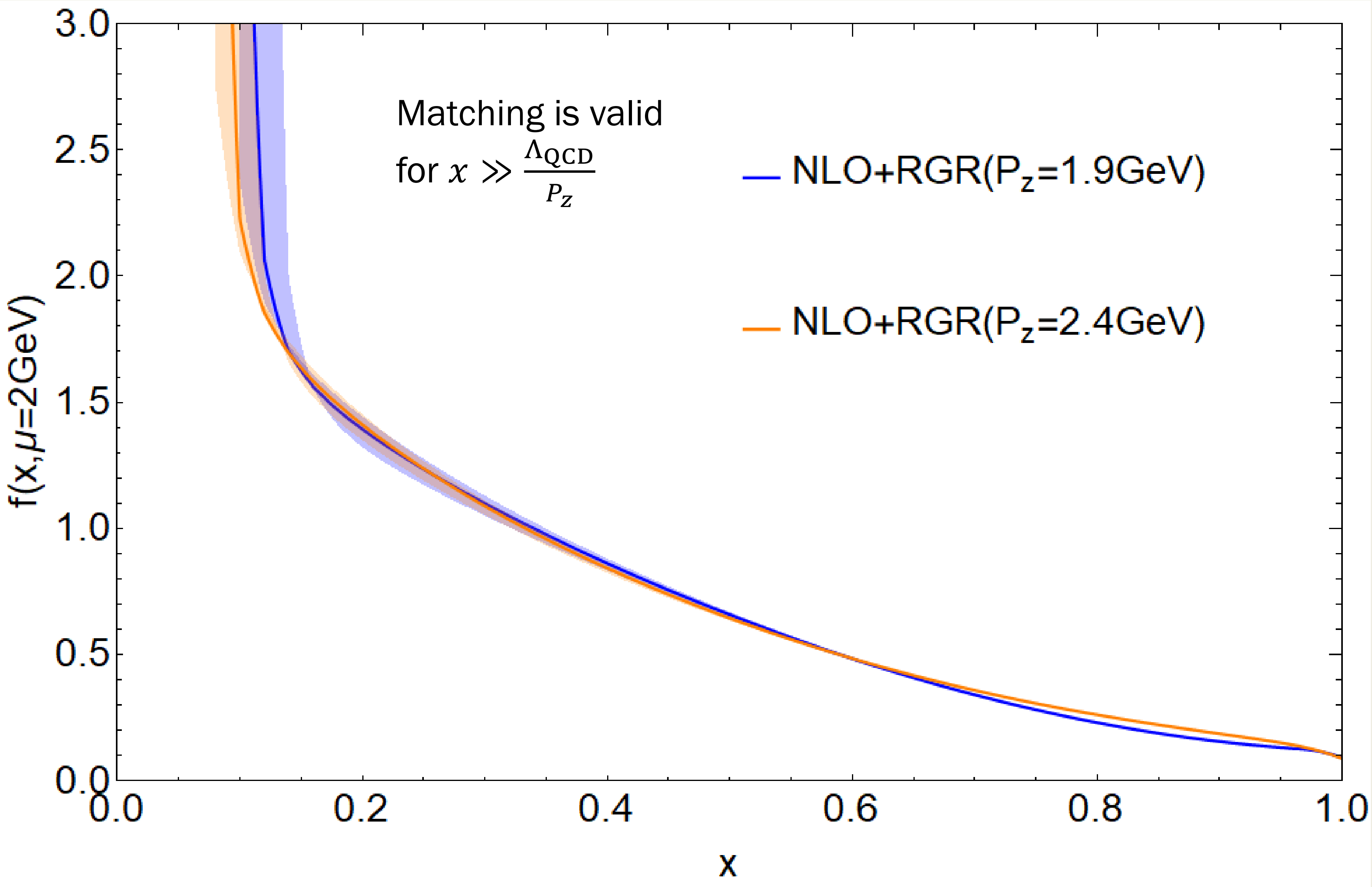
NLO NNLO

$$1 + \alpha_s(\mu)[c_{11}L + c_{10}] + \alpha_s^2(\mu)[c_{22}L^2 + c_{21}L + c_{20}] + \alpha_s^3(\mu)[c_{33}L^3 + c_{32}L^2 + c_{31}L + c_{30}] + \cdots$$

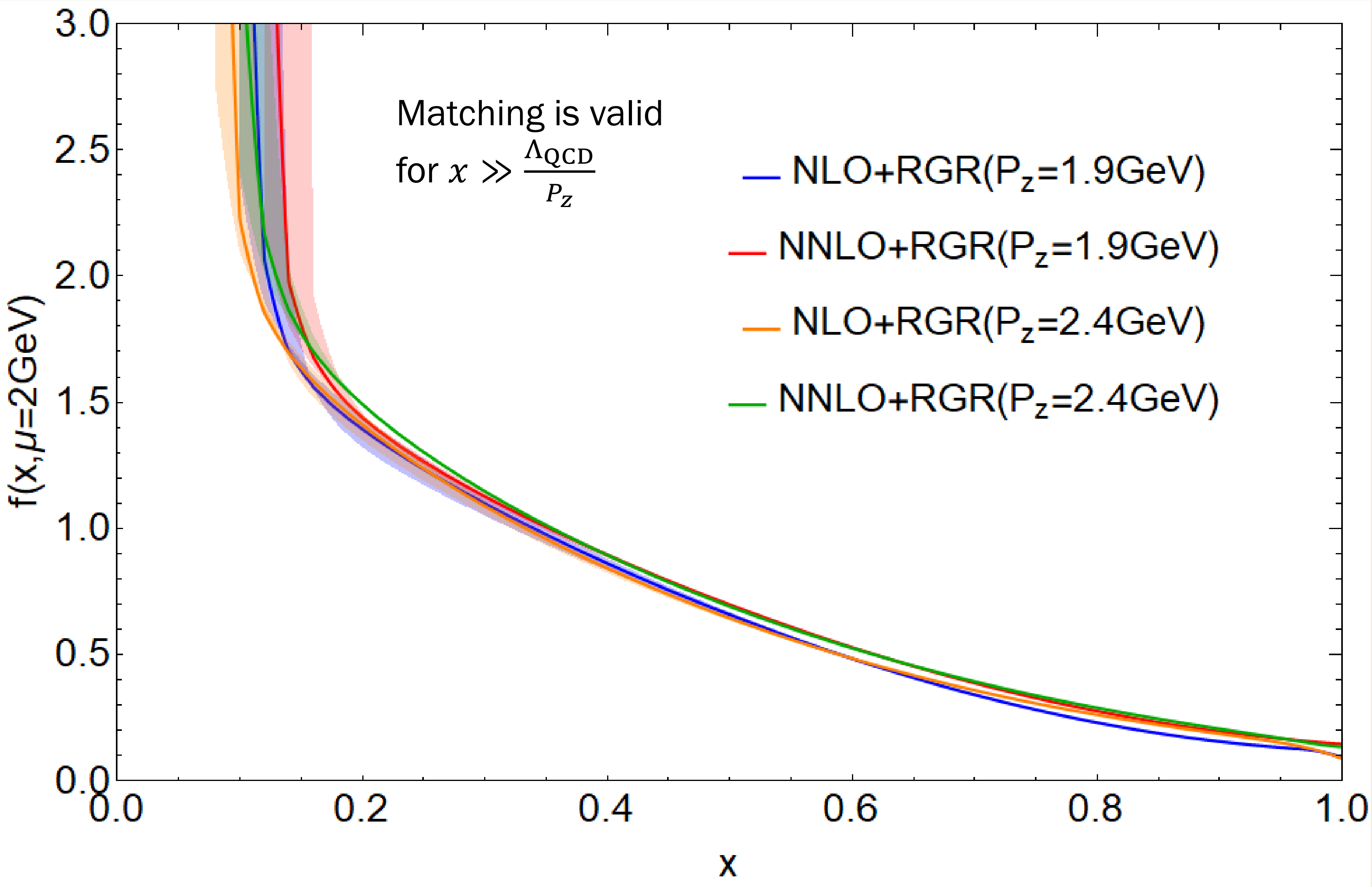
NLO+RGR NNLO+RGR

$$C_{\text{RGR}}^{-1}\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right) = e^{P\cdots}\left[\delta\left(1-\frac{x}{y}\right) + \alpha_s(2xP_z c') * \cdots\right] \qquad c' = 0.8\sim1.2$$

Compare different momenta



Compare different momenta



Conclusions

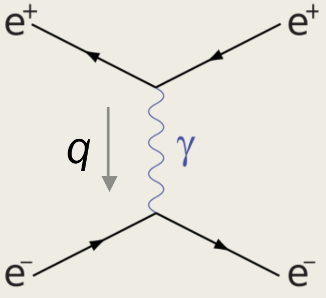
- Resummation of small-momentum large logarithms improves the prediction accuracy at intermediate x
- RGR exposes the breakdown of perturbation theory at small x, which is consistent with the conclusion drawn from higher-twist power counting
- Small x can be improved if we increase the momentum, which is made clear by RGR matching

Appendix

Resum large logarithms to recover the intrinsic physical scale

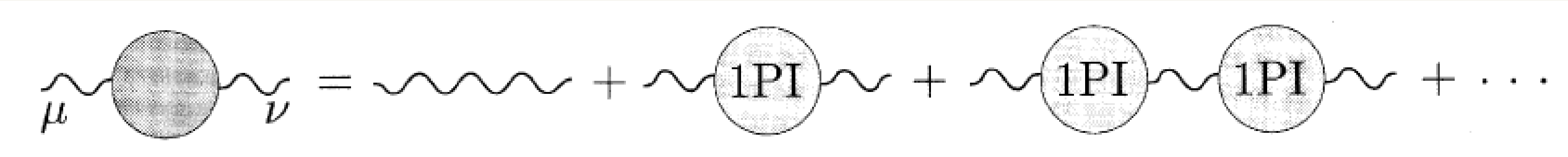
- QED example:

  Bhabha scattering $e^+e^- \to e^+e^-$



$$\sim \alpha(\mu) \sim \alpha(m_e)$$

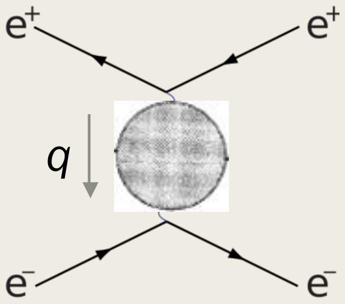
- Resummation



$$\sim \frac{1}{1-\frac{\alpha(\mu)}{3\pi}ln[\frac{-q^2}{\mu^2}]}$$

$$\frac{\alpha(\mu)}{3\pi}ln[\frac{-q^2}{\mu^2}] \qquad \left(\frac{\alpha(\mu)}{3\pi}ln[\frac{-q^2}{\mu^2}]\right)^2$$

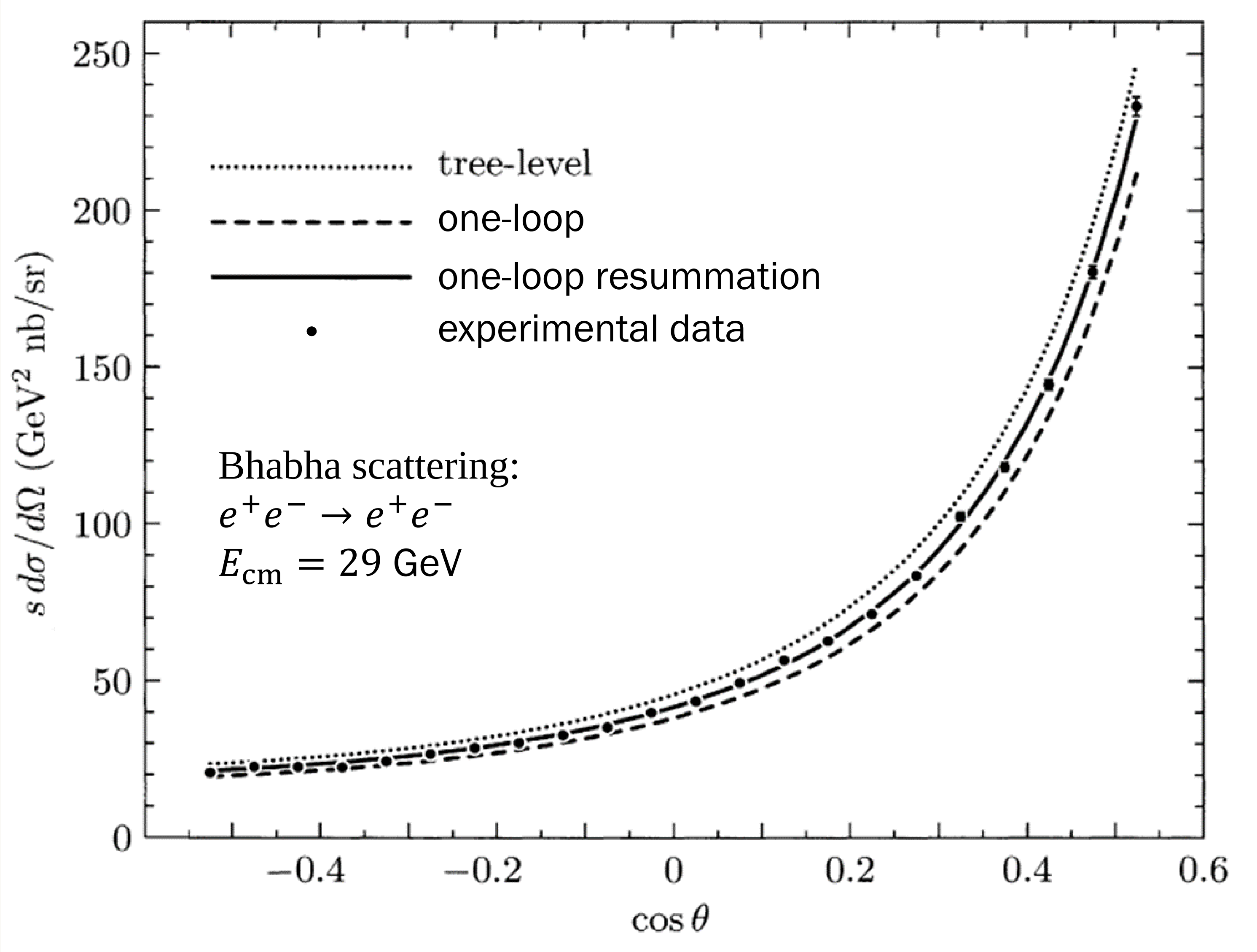
If $-q^2 = (29\ \mathrm{GeV})^2, \mu^2 = (0.5\ \mathrm{MeV})^2$, $\ln\left[\frac{-q^2}{\mu^2}\right] = 22$



$$\sim \frac{\alpha(\mu)}{1-\frac{\alpha(\mu)}{3\pi}ln[\frac{-q^2}{\mu^2}]} \sim \alpha(\sqrt{-q^2})$$

Peskin&Shroeder QFT

Resummation improves prediction accuracy



Peskin&Shroeder QFT

Normalization in Hybrid scheme

- $\overline{\text{MS}}$ scheme matching kernel

$$C^{\overline{\text{MS}}}\left(\xi,\frac{\mu}{|x|P_z}\right)=\delta(1-\xi)+\frac{\alpha_s(\mu)C_F}{2\pi}\begin{cases}\left(\frac{1+\xi^2}{1-\xi}\ln\left(\frac{\xi}{\xi-1}\right)+1-\frac{3}{2(1-\xi)}\right)_{+(1)}^{[1,\infty]} & \xi>1\\ \left(\frac{1+\xi^2}{1-\xi}\left[-L+\ln\left(\frac{1-\xi}{\xi}\right)-1\right]+1+\frac{3}{2(1-\xi)}\right)_{+(1)}^{[0,1]} & 0<\xi<1\\ \left(-\frac{1+\xi^2}{1-\xi}\ln\left(\frac{-\xi}{1-\xi}\right)-1+\frac{3}{2(1-\xi)}\right)_{+(1)}^{[-\infty,0]} & \xi<0\end{cases}+\frac{\alpha_s(\mu)C_F}{2\pi}\delta(1-\xi)\left(\frac{3}{2}L+\frac{5}{2}\right)$$

- Hybrid renormalized matrix element is scale independent:

$$\tilde{h}(z,P_z)=\frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{\tilde{h}^{\text{lat}}(z,a,0)}\theta(z_s-|z|)+\frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{Z^R(z,a,\mu)\tilde{h}^{\overline{\text{MS}}}(z_s,\mu,0)}\theta(|z|-z_s)$$

- Heavy light quark term is eliminated in the matching kernel,

  e.g. NLO Hybrid scheme matching kernel

$$C^{\text{hybrid}}\left(\xi,\frac{\mu}{|x|P_z}\right)$$

$$=C^{\overline{\text{MS}}}\left(\xi,\frac{\mu}{|x|P_z}\right)-\frac{\alpha_s(\mu)C_F}{2\pi}\delta(1-\xi)\left(\frac{3}{2}L+\frac{5}{2}\right)+\frac{\alpha_s(\mu)C_F}{2\pi}\frac{3}{2}\left[-\frac{1}{|1-\xi|}+\frac{2\text{Si}[(1-\xi)|y|z_sP_z]}{\pi(1-\xi)}\right]_{+(1)}^{[-\infty,\infty]}$$

The physics scale after inversion

- Before the inversion, the physics scale $2xP_z$ is for quasi parton

$$\tilde{f}(x, P_z) = C\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right) \otimes f(y, \mu) + \mathcal{O}\left[\frac{\Lambda_{\mathrm{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(1-x)^2 P_z^2}\right]$$

- Iterative inversion

$$\tilde{f}(x, P_z) = \left[\delta\left(1 - \frac{x}{y}\right) + \delta C\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right)\right] \otimes f(y, \mu)$$

$$\tilde{f}(x, P_z) = f(x, \mu) + \delta C\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right) \otimes f(y, \mu)$$

$$f(x, \mu) = \tilde{f}(x, P_z) - \delta C\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right) \otimes f(y, \mu)$$

$$f(x, \mu) = \tilde{f}(x, P_z) - \delta C\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right) \otimes \tilde{f}(y, P_z) + \delta C \otimes \delta C \otimes \tilde{f} - \delta C \otimes \delta C \otimes \delta C \otimes \tilde{f} + \cdots$$

- After the inversion, the physics scale $2xP_z$ is for light cone parton

$$f(x, \mu) = C^{-1}\left(\frac{x}{y}, \frac{\mu}{|x|P_z}\right) \otimes \tilde{f}(y, P_z) + \mathcal{O}\left[\frac{\Lambda_{\mathrm{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(1-x)^2 P_z^2}\right]$$

An equivalent approach for the RGR matching

- RGR Matching formula

$$f = C_{\mathrm{RGR}}^{-1} \otimes \tilde{f} = M\{e^{P}\} \otimes C^{-1} \otimes \tilde{f}$$

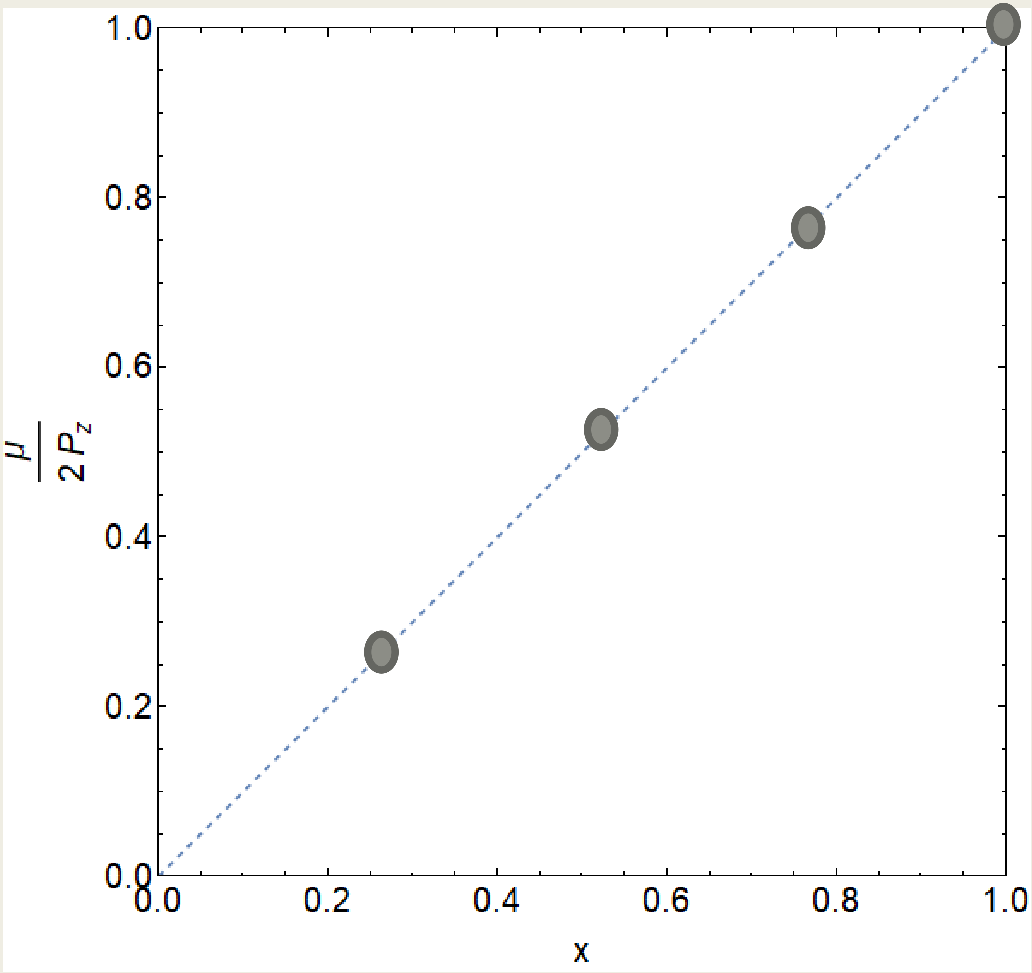
- Two Steps for RGR matching

1. Fixed order matching at physics scale:

$$f(x, 2xP_z) = C^{-1}\left(\frac{x}{y}, \frac{2xP_z}{|x|P_z}\right) \otimes \tilde{f}(y, P_z)$$

2. DGLAP evolve $f(x, 2xP_z)$ to $f(x, \mu)$:

$$\mu \frac{df(x,\mu)}{d\mu} = \int_x^1 \frac{dw}{w} P[w, \alpha_s(\mu)] f\left(\frac{x}{w}, \mu\right)$$



Compare RGR matching with fixed order one