# Progress in calculation of the fourth Mellin moment of the pion LCDA using the HOPE method

THE HOPE COLLABORATION

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with
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# Takeaways

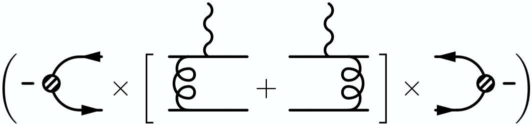
- Light cone distribution amplitude (LCDA) important in factorization theorem for pion electromagnetic form factor.
- Heavy-quark Operator Product Expansion (HOPE) $\rightarrow$ moments of LCDA.
- Preliminary results for $\left\langle \xi^4 \right\rangle$ at range of finite lattice spacings, twist-suppressions.

## Factorization for $F_\pi(Q^2)$: 1979

- $\phi_M(x, \mu^2)$: Light Cone Distribution Amplitude (LCDA): Probability amplitude
- Factorization theorem:

$$F_\pi(Q^2) \underset{\text{large } Q^2}{=} \int_0^1 dxdy \phi_{\overline{M}}(y, Q^2) T_H(x, y, Q^2) \phi_M(x, Q^2)$$

$$\underset{\text{large } Q^2}{=} \int_0^1 dxdy$$



$$\underset{\text{large } Q^2}{=} \frac{16\pi\, \alpha_S(Q^2)}{Q^2} f_\pi^2 \omega_\phi^2(Q^2) \to \frac{16\pi\, \alpha_S(Q^2)}{Q^2} f_\pi^2$$

$$\omega_\phi(Q^2) = \frac{1}{3} \int_0^1 dx \, \frac{\phi(x, Q^2)}{x}$$

# $F_\pi(Q^2)$

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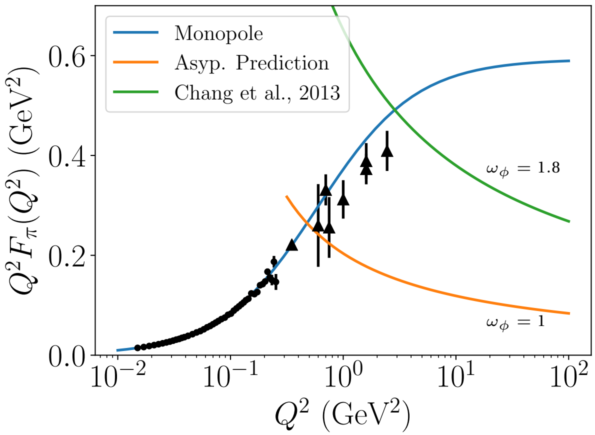
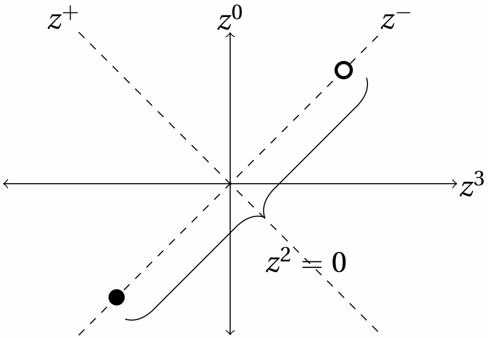



Figure 1: Data from Amendolia et al. (1986), Huber et al. (2008)

# Calculating the Pion LCDA

- Only *ab-initio* method to calculate non-perturbative QCD: Lattice QCD.
- Problem: LCDA defined as

$$\langle \Omega \left| \bar{\psi}(z_-)\gamma_\mu\gamma_5 W[z_-, -z_-]\psi(-z_-) \right| \pi(\mathbf{p}) \rangle = i p_\mu f_\pi \int_{-1}^{1} d\xi \mathrm{e}^{i\xi p^+ \cdot z_-} \phi_\pi(\xi, \mu^2)$$



# Options

- Calculate Mellin moments directly:

$$\langle \xi^n \rangle = \int_{-1}^{1} d\xi \, \xi^n \phi_\pi(\xi, \mu^2)$$

  - G. S. Bali et al., JHEP 2019.
  - V. M. Braun, et al., PRD 2015.
- Utilize Factorization Theorem
  - X. Ji, PRL 2013.
  - A. V. Radyushkin, PRD 2017.
  - Ma, Y.-Q., Qiu, J.-W. PRD, 2018.
- Match hadronic matrix element to OPE
  - V. Braun and D. Müller, EPJC 2008.
  - W. Detmold and C. J. D. Lin, PRD 2006.
  - Chambers et al, PRL 2017

# Heavy-quark OPE Detmold & Lin, 2006

- Consider matrix element

$$V^{\mu\nu}(p,q) = \int d^4 z e^{iq\cdot z} \langle\Omega| T\{J^\mu_\Psi(z/2) J^\nu_\Psi(-z/2)\} |\pi(\mathbf{p})\rangle$$

$$J^\mu_\Psi = \overline{\Psi}(x)\Gamma^\mu\psi(x) + \overline{\psi}(x)\Gamma^\mu\Psi(x)$$

- Perform operator product expansion:

$$\tilde{Q}^2 = -q^2 + m_\Psi^2 \qquad \text{large scale}$$

$$\tilde{\omega} = \frac{1}{\tilde{x}} = \frac{2p\cdot q}{\tilde{Q}^2} \qquad \text{expansion parameter}$$

$$V^{\mu\nu}_{\text{HOPE}}(p,q) = K[1 + \tilde{\omega}^2 \langle\xi^2\rangle + \tilde{\omega}^4 \langle\xi^4\rangle + \dots] + \underbrace{\mathcal{O}(\alpha_S)}_{\text{Perturbative corrections}} + \underbrace{\mathcal{O}(1/Q^3)}_{\text{Higher twist}}$$

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# Overview of Calculation

- Pion LCDA $\phi_\pi(\xi, \mu^2)$ important in description of $F_\pi(Q^2)$
- Long-range sensitive: non-perturbative.
- HOPE method allows for determination of moments:

$$\langle \xi^n \rangle (\mu^2) = \int_{-1}^{1} d\xi \, \xi^n \phi(\xi, \mu^2)$$

- HOPE Method:

$$V^{\mu\nu}_{\mathsf{LQCD}}(p, q; a) = \int d^4 z e^{iq\cdot z} \langle \Omega | \, T\{J^\mu_\Psi(z/2) J^\nu_\Psi(-z/2)\} \, |\pi(\mathbf{p})\rangle$$

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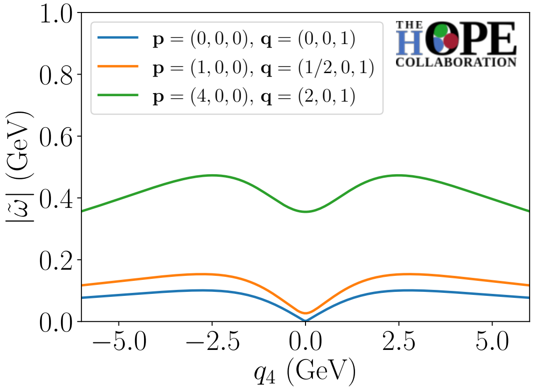
# Optimizing Kinematics

$$V^{\mu\nu}(p,q) = K[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \ldots], \qquad \tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2}$$

- Choose $\mathbf{p} = (2,0,0) \times 2\pi/L$



# Lattice Details

| $L^3 \times T$ | $a$ (fm) | $N_{\text{cfg}}$ | $N_{\Psi}$ |
|---|---|---|---|
| $24^3 \times 48$ | 0.0813 | 6500 | 4 |
| $32^3 \times 64$ | 0.0600 | 4500 | 3 |
| $40^3 \times 80$ | 0.0502 | $\mathcal{O}(5000)$ | 4 |
| $48^3 \times 96$ | 0.0407 | $\mathcal{O}(5000)$ | 5 |

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- Wilson-clover fermions with non-perturbatively tuned $c_{\text{SW}}$
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Still to come...

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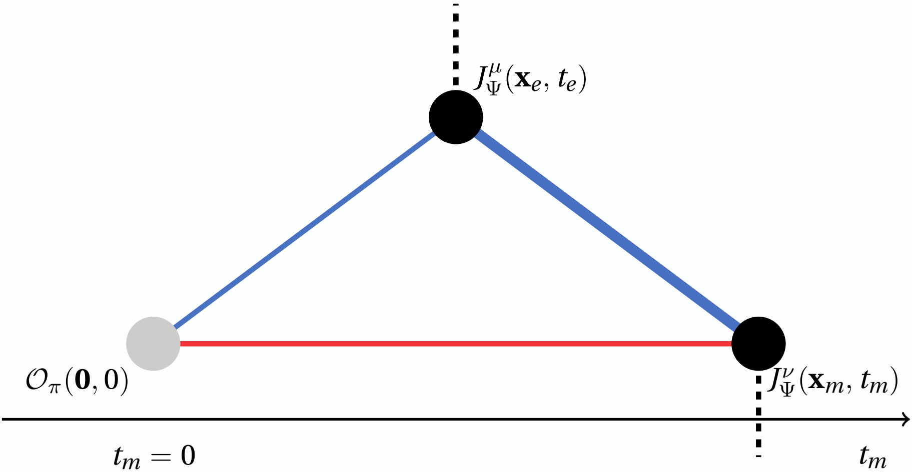
# Matrix Element Calculated

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = \int d^3x_e d^3x_m e^{i\mathbf{p}_e\cdot\mathbf{x}_e + i\mathbf{p}_m\cdot\mathbf{x}_m} \langle\Omega| \, T\{J_\Psi^\mu(x_e) J_\Psi^\nu(x_m) \mathcal{O}_\pi^\dagger(0)\} \, |\Omega\rangle$$



# Ratio Method

- Excited state dependent only on sum $t_e + t_m$.

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q}) \frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2} + \dots,$$

- Define $t_+ = t_e + t_m$, $t_- = t_e - t_m$.

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = R^{\mu\nu}(t_-; \mathbf{p}, \mathbf{q}) \frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})t_+/2} + \dots,$$

- Consider two sets of time; $(t_e, t_m)$ and $(t'_e, t'_m) = (t_e + \delta, t_m - \delta)$

$$t'_+ = (t_e + \delta) + (t_m - \delta) = t_e + t_m = t_+$$
$$t'_- = (t_e + \delta) - (t_m - \delta) = t_e - t_m + 2\delta \neq t_-$$

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- Consider two sets of time; $(t_e, t_m)$ and $(t_e', t_m') = (t_e + \delta, t_m - \delta)$

$$\begin{aligned} t_+' &= (t_e + \delta) + (t_m - \delta) = t_e + t_m = t_+ \\ t_-' &= (t_e + \delta) - (t_m - \delta) = t_e - t_m + 2\delta \neq t_- \end{aligned}$$

# Ratio Method

- Construct ratio with fixed $t_+$, varying $t_-$ $(\delta = -1)$

$$\mathcal{R} = \frac{C_3^{\mu\nu}(t_e - 1, t_m + 1; \mathbf{p}_e, \mathbf{p}_m)}{C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m)}$$

$$= \frac{R^{\mu\nu}(t_e - t_m - 2; \mathbf{p}, \mathbf{q})}{R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q})} \frac{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}}{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}} \left[1 + \dots\right]$$

- Need two $t_e$, ie $t_e$ and $t_e - 1$
- No need for 2-point data!
- No renormalization required.
- No $f_\pi$

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# Operator Optimization

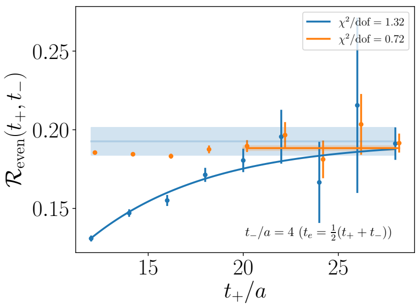
- Momentum smearing (Bali et al)
- Variational analysis:

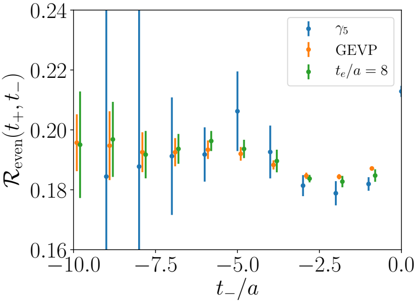
$$\mathcal{O}_\pi(x) = c_1\mathcal{O}_1(x) + c_2\mathcal{O}_2(x), \quad \mathcal{O}_1(x) = \overline{\psi}\gamma_5\psi, \ \ \mathcal{O}_2(x) = \overline{\psi}\gamma_4\gamma_5\psi$$

$$\begin{aligned} C^{\mu\nu}_{3,\mathsf{GEVP}}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) &= c_1 C^{\mu\nu}_{3,\gamma_5}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) \\ &+ c_2 C^{\mu\nu}_{3,\gamma_4\gamma_5}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) \end{aligned}$$

# Excited State Contamination

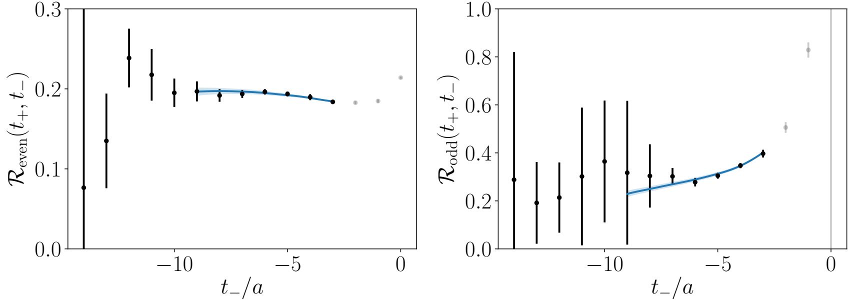
$$\mathcal{R} == \frac{R^{\mu\nu}(t_e - t_m - 2; \mathbf{p}, \mathbf{q})}{R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q})} \cancel{\frac{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}}{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}}} \left[1 + \dots\right]$$



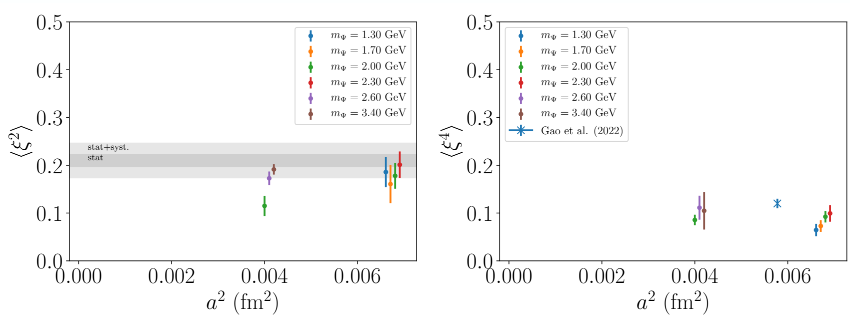
# Example at fixed lattice spacing

$$\mathcal{R}(t_-, \mathbf{p}, \mathbf{q}, m_\psi, \langle \xi^2 \rangle, \langle \xi^2 \rangle; a) = \mathcal{R}_{\mathsf{HOPE}}(t_-, \mathbf{p}, \mathbf{q}, m_\psi, \langle \xi^2 \rangle, \langle \xi^2 \rangle) + \mathcal{O}(a^2)$$




$L/a = 24$, $m_\Psi = 2.0$ GeV, $\langle \xi^2 \rangle = 0.17 \pm 0.04$, $\langle \xi^4 \rangle = 0.07 \pm 0.02$

# Status of Calculation




$$\langle \xi^n \rangle (a, m_\Psi) = \langle \xi^n \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2$$

# Further Work & Conclusions

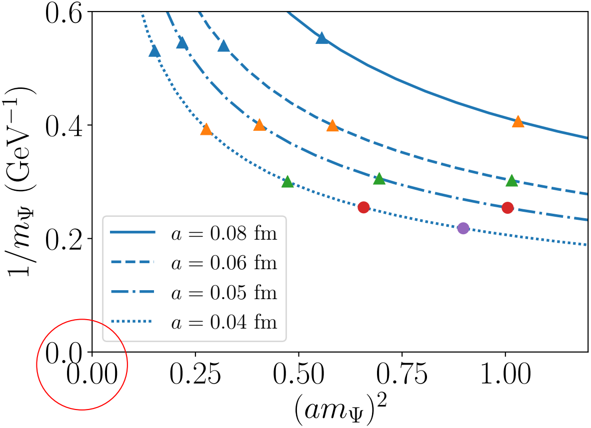
- HOPE can be used to extract Mellin moments.
- Introduced ratio method: no renormalzation.
- 2 lattice spacings, 3/4 heavy quark masses.
- Aim for 2 more lattice spacings + more heavy quark masses to enable continuum, twist-2 extrapolation.

# Higher-twist/Lattice Artifacts tradeoff

Larger twist artifacts

Smaller twist artifacts

$$m_\Psi \lessapprox \frac{1}{a}$$

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