Interpolation method from instant form to the light-front dynamics

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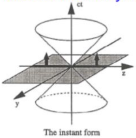
2022 Meeting on Lattice Parton Physics from Large-Momentum Effective Theory (LaMET 2022)

Outline

The instant form and light front form: Dirac's proposition

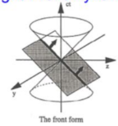
IFD

Instant Form Dynamics



LFD

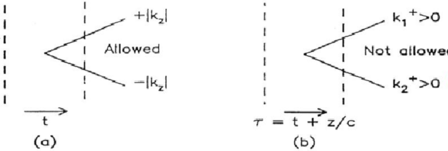
Light-Front Dynamics



Energy-Momentum Dispersion Relations

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2} \qquad p^- = \frac{p_\perp^{\,2} + m^2}{p^+}$$

Time-ordered Diagrams: Vacuum fluctuations



Except zero-modes

$$k_1^+ = k_2^+ = 0$$

The interpolation coordinates between the IFD and LFD

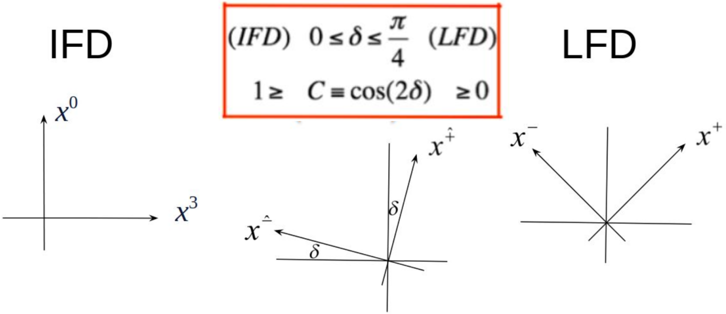
We define interpolating space-time coordinates

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

With the short-hand notation $\mathbb{S} = \sin 2\delta$ and $\mathbb{C} = \cos 2\delta$, we have the metric written as

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}.$$

The Interpolation coordinates between the IFD and LFD



The interpolation coordinates between the IFD and LFD

We also have

$$a^{\hat{\mu}} b_{\hat{\mu}} = \left(a_{\hat{+}} b_{\hat{+}} - a_{\hat{-}} b_{\hat{-}}\right) \mathbb{C} + \left(a_{\hat{+}} b_{\hat{-}} + a_{\hat{-}} b_{\hat{+}}\right) \mathbb{S} - a_{\hat{1}} b_{\hat{1}} - a_{\hat{2}} b_{\hat{2}}$$

So the four-momentum squared written explicitly in the interpolation coordinates becomes

$$p^{\hat{\mu}} p_{\hat{\mu}} = p_{\hat{+}}^2 \mathbb{C} - p_{\hat{-}}^2 \mathbb{C} + 2 p_{\hat{+}} p_{\hat{-}} \mathbb{S} - \mathbf{p}_{\perp}^2$$

or it can be re-written as

$$p^{\hat{\mu}} p_{\hat{\mu}} = \frac{p^{\hat{+}2}}{\mathbb{C}} - \frac{p_{\hat{-}}^2}{\mathbb{C}} - \mathbf{p}_{\perp}^2 \equiv p'^{\hat{+}2} - p'^2_{\hat{-}} - \mathbf{p}_{\perp}^2$$

The interpolation Hamiltonian dynamics between the IFD and LFD

K. Hornbostel, Phys. Rev. D 45, 3781 (1992)

The Lagrangian for a free massive scalar theory in 1+1 dimensions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

in the interpolation coordinates becomes

$$\mathcal{L} = \frac{1}{2}\mathbb{C}\left[\left(\partial_{\hat{+}}\phi\right)^2 - \left(\partial_{\hat{-}}\phi\right)^2\right] + \mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \frac{1}{2}m^2\phi^2,$$

with the corresponding equation of motion

$$\left[\partial^2 + m^2\right]\phi = \left[\mathbb{C}\left(\partial_{\hat{+}}^2 - \partial_{\hat{-}}^2\right) + 2\mathbb{S}\partial_{\hat{+}}\partial_{\hat{-}} + m^2\right]\phi = 0.$$

The interpolation Hamiltonian dynamics between the IFD and LFD

Plane-wave expansion of this scalar field becomes

$$\phi(x) = \int \frac{dp_{\hat{-}}}{2\pi\sqrt{2\omega_p}} \left[a(p_{\hat{-}})e^{-i(p_{\hat{+}}x^{\hat{+}}+p_{\hat{-}}x^{\hat{-}})} + a^\dagger(p_{\hat{-}})e^{i(p_{\hat{+}}x^{\hat{+}}+p_{\hat{-}}x^{\hat{-}})}\right]$$

with the energy $p_{\hat{+}}$ given by

$$p_{\hat{+}} = \frac{\omega_p - \mathbb{S}p_{\hat{-}}}{\mathbb{C}},$$

with

$$\omega_p = \sqrt{p_{\hat{-}}^2 + \mathbb{C}m^2}.$$

The equal-interpolation-time Hamiltonian dynamics is defined by imposing the quantization condition

$$\left[a(p_{\hat{-}}), a^\dagger(p'_{\hat{-}})\right]_{x^{\hat{+}}=x'^{\hat{+}}} = \delta(p_{\hat{-}} - p'_{\hat{-}}).$$

The interpolation Hamiltonian dynamics between the IFD and LFD

The Hamiltonian, which is conjugate to the time $x^{\hat{+}}$, is

$$P_{\hat{+}} = \int dp_{\hat{-}} \left[\frac{\omega_p - \mathbb{S}p_{\hat{-}}}{\mathbb{C}}\right] a^\dagger(p_{\hat{-}})a(p_{\hat{-}}),$$

while the momentum is

$$P_{\hat{-}} = \int dp_{\hat{-}} \left[p_{\hat{-}}\right] a^\dagger(p_{\hat{-}})a(p_{\hat{-}}).$$

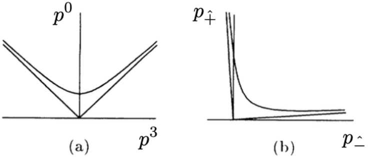


FIG. **Energy vs momentum at (a) equal time and (b) near the light cone for both massive and massless particles.**

The interpolation scattering amplitudes

PHYSICAL REVIEW D **87**, 065015 (2013)

Interpolating scattering amplitudes between the instant form and the front form of relativistic dynamics

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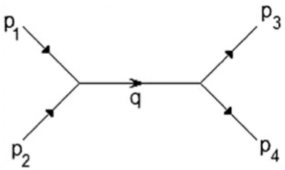
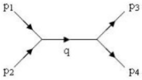
(Received 7 December 2012; published 19 March 2013)



$$\Sigma = \frac{1}{s - m^2}$$

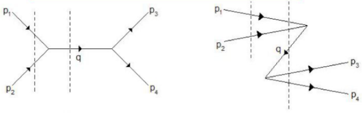
The interpolation scattering amplitudes



$$\delta = 0 \qquad p_0 = p^0 \qquad -p_3 = p^3$$

$$0 < \delta < \pi/4 \qquad p_{\hat{+}} = p^0 \cos\delta - p^3 \sin\delta \qquad p_{\hat{-}} = p^0 \sin\delta + p^3 \cos\delta$$

$$\delta = \pi/4 \qquad p_+ = p^- \qquad p_- = p^+$$



(a) (b)

$$\frac{1}{2q^0}\left(\frac{1}{p_1^0 + p_2^0 - q^0} - \frac{1}{p_1^0 + p_2^0 + q^0}\right) \leftarrow \frac{1}{2\omega_q}\left(\frac{1}{P_{\hat{+}} + \frac{\mathbb{S}q_{\hat{-}} - \omega_q}{\mathbb{C}}} - \frac{1}{P_{\hat{+}} + \frac{\mathbb{S}q_{\hat{-}} + \omega_q}{\mathbb{C}}}\right) \rightarrow \frac{1}{P^+}\frac{1}{\left\{P^- - \frac{(\vec{P}_\perp^2 + m^2)}{2P^+}\right\}}$$

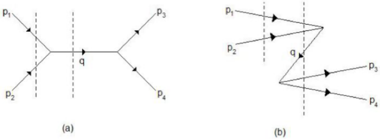
$$\omega_q = \sqrt{q_{\hat{-}}^2 + \mathbb{C}\left(\vec{q}_\perp^2 + m^2\right)}$$

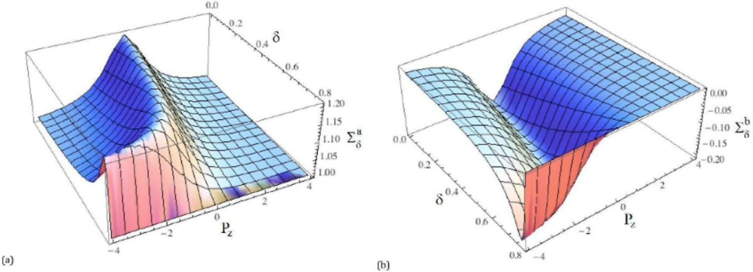
$$\mathbb{C} = \cos 2\delta$$

$$\mathbb{S} = \sin 2\delta$$

$$\frac{\mathbb{S}q_{\hat{-}} + \omega_q}{\mathbb{C}} \to \frac{2}{\mathbb{C}} - \frac{\vec{q}_\perp^2 + m^2}{2q_{\hat{-}}} + \mathcal{O}(\mathbb{C}) \to \infty \quad \text{as } \mathbb{C} \to 0$$

The interpolation scattering amplitudes



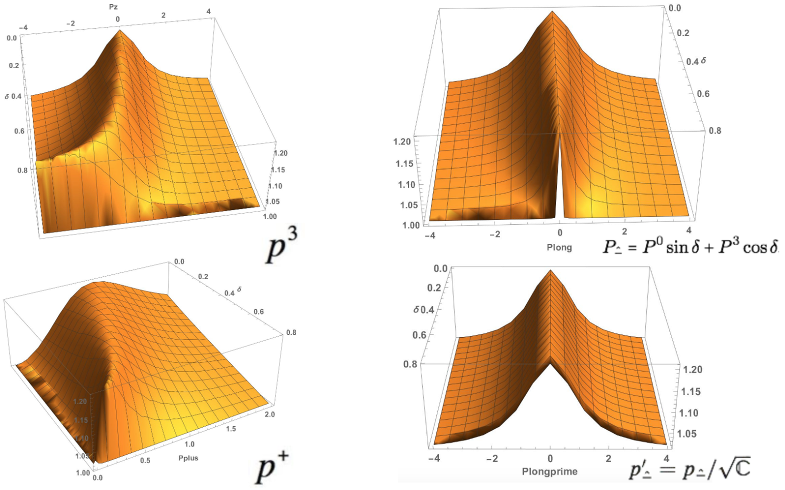


$\Sigma(a)+\Sigma(b)=1/(s-m^2)$; s=2 GeV^2, m=1GeV

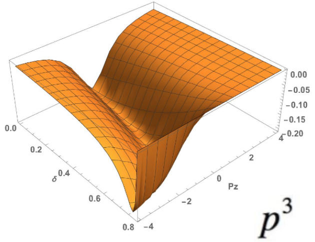
J-shape peak & valley : $P_z = -\sqrt{\dfrac{s(1-C)}{2C}}$; $C = \cos(2\delta)$

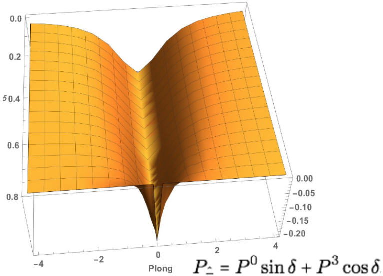
As C → 0, $P^+ = P^0+P_z$ →0 leads to LF Zero-modes.

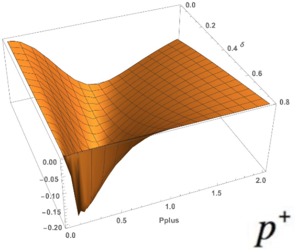
The interpolation scattering amplitudes

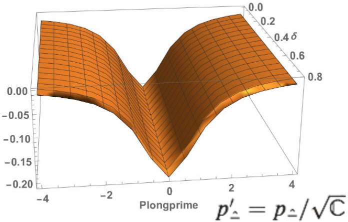


The interpolation scattering amplitudes







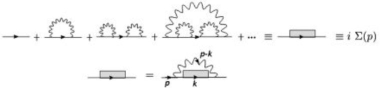


'T Hooft model

BM and C.-R Ji, Phys. Rev. D 104, 036004 (2021)

- ★ 1 space 1 time dimensions, where confinement arises naturally due to the linear potential.
- ★ Large N_c ('t Hooft coupling $\lambda \sim g^2 N_c$ is kept finite while $N_c \rightarrow \infty$ and $g \rightarrow 0$), so that non-planar diagrams are negligible.

Mass gap equation



$$\Sigma(p_{\perp}) = \sqrt{\mathbb{C}} A(p_{\perp}) + \gamma_{\perp} B(p_{\perp})$$

$$E(p_{\perp}) \cos\theta(p_{\perp}) = \sqrt{\mathbb{C}} m + \mathbb{C} \cdot \frac{\lambda}{2} \int \frac{dk_{\perp}}{(p_{\perp} - k_{\perp})^2} \cos\theta(k_{\perp})$$

$$E(p_{\perp}) \sin\theta(p_{\perp}) = p_{\perp} + \mathbb{C} \cdot \frac{\lambda}{2} \int \frac{dk_{\perp}}{(p_{\perp} - k_{\perp})^2} \sin\theta(k_{\perp})$$

'T Hooft model

Bound state equation

$$\Gamma(r,p) = \frac{i\lambda}{2\pi}\int \frac{dk_- dk_+}{(p_- - k_-)^2} S(p)\gamma^+\Gamma(r,k)\gamma^+ S(p-r)$$

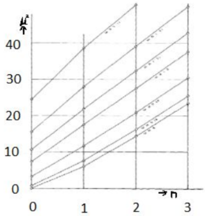
$$\left[-r_+ + \frac{-Sp_- + E(p_-)}{C} + \frac{S(p_- - r_-) + E(p_- - r_-)}{C}\right]\hat\phi_+(r_-,p_-)$$
$$= \lambda ⨍ \frac{dk_-}{(p_- - k_-)^2}\left[C(p_-,k_-,r_-)\hat\phi_+(r_-,k_-) - S(p_-,k_-,r_-)\hat\phi_-(r_-,k_-)\right],$$
$$\left[r_+ + \frac{-S(p_- - r_-) + E(p_- - r_-)}{C} + \frac{Sp_- + E(p_-)}{C}\right]\hat\phi_-(r_-,p_-)$$
$$= \lambda ⨍ \frac{dk_-}{(p_- - k_-)^2}\left[C(p_-,k_-,r_-)\hat\phi_-(r_-,k_-) - S(p_-,k_-,r_-)\hat\phi_+(r_-,k_-)\right].$$

LFD $$\left[\mathcal{M}^2 - \frac{m^2 - 2\lambda}{x} - \frac{m^2 - 2\lambda}{1-x}\right]\phi(x) = -2\lambda ⨍_0^1 \frac{dy}{(x-y)^2}\phi(y)$$

'T Hooft model

Meson spectroscopy

LFD



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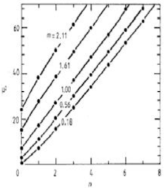
IFD



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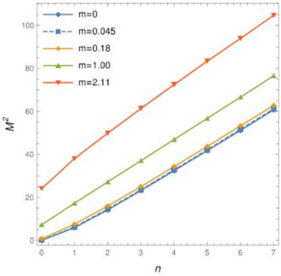
JHEP, 11:151.

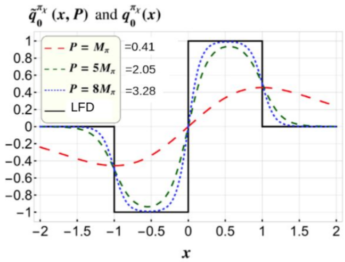
Interpolation form



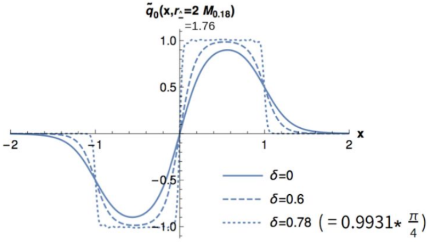
Ma, B. and Ji, C.-R. (2021).
Phys. Rev. D, 104:036004.

'T Hooft model

Quasi-PDF



Quark quasi-PDFs and light-front PDF for the chiral pion.



Interpolating "quasi-PDFs" for the chiral pion.

All quantities are in proper units of $\sqrt{2\lambda}$.

Jia, Y., Liang, S., Xiong, X., and Yu, R. (2018). Phys. Rev. D, 98:054011.

Ma, B. and Ji, C.-R. (2021). Phys. Rev. D, 104:036004.

Outlook

Extended Wick Rotation

$$p^0 \to \tilde{P}^0 = ip^0 \quad (\delta = 0)$$

For $0 < \delta < \pi/4$,

$$p^{\hat{+}}/\sqrt{C} \to \tilde{P}^{\hat{+}}/\sqrt{C} = ip^{\hat{+}}/\sqrt{C}.$$

Correspondence to Euclidean Space

$$p'^2_{\hat{-}} = p^2_{\hat{-}}/C \leftrightarrow -\tilde{P}^2$$

Thank you for your attention!