Continuum limit of nucleon quasi-PDFs

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How to determine PDFs

Phenomenological PDFs: need

1. Factorization expression (perturbation theory)
2. Cross section (experiment)
3. Suppression of higher twist (high Q^2)

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New lattice methods: need

1. Factorization expression (perturbation theory)
2. Bilocal matrix element (lattice) $\sim \langle p_z|\phi(z)\phi(0)|p_z\rangle$
3. Suppression of higher twist (high p_z or small z)

(Quasi-)PDFs

Consider bilocal matrix element

$$f(x) = \int \frac{dz}{4\pi} e^{-ixz\mathbf{n}\cdot\mathbf{p}} \left\langle \mathbf{p} \middle| \bar{\psi}(z\mathbf{n}) \not{n} U(z\mathbf{n}, 0)\psi(0) \middle| \mathbf{p} \right\rangle$$

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- $\mathbf{n} = \hat{z} \implies f(x) = \tilde{q}(x, p_z)$ quasi-PDF

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Factorization:

$$\tilde{q}(x, p_z; \mu) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y}, \frac{\mu}{p_z}\right) q(y; \mu) + O\left(\frac{1}{p_z^2}\right)$$

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Many systematics in lattice calculation:
excited states, finite volume, discretization effects, higher twist

Outline

1. $O(a)$ effects from nonlocal operator
   JRG, K. Jansen, F. Steffens, Phys. Rev. Lett. **121**, 022004 (2018) [1707.07152]
   JRG, K. Jansen, F. Steffens, Phys. Rev. D **101**, 074509 (2020) [2002.09408]
2. Continuum limit study
   C. Alexandrou, K. Cichy, M. Constantinou, JRG, K. Hadjiyiannakou, K. Jansen, F. Manigrasso, A. Scapellato, F. Steffens, Phys. Rev. D (to appear) [2011.00964]

Symanzik approach

Use an effective field theory to describe lattice QCD at $a > 0$.

$$\mathcal{L} = \mathcal{L}^{(0)} + a\mathcal{L}^{(1)} + a^2\mathcal{L}^{(2)} + \cdots$$
$$\mathcal{O} = \mathcal{O}^{(0)} + a\mathcal{O}^{(1)} + a^2\mathcal{O}^{(2)} + \cdots$$

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E.g. Wilson action:

$$\mathcal{L}^{(0)} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}(\not{D} + m_R)\psi, \qquad \mathcal{L}^{(1)} = c\,\bar{\psi} i\sigma_{\mu\nu} F_{\mu\nu}\psi + \cdots$$

Axial current:

$$\mathcal{O}^{(0)} = \bar{\psi}\gamma_\mu\gamma_5\tau^a\psi, \qquad \mathcal{O}^{(1)} = c'\,\partial_\mu\left(\bar{\psi}\gamma_5\tau^a\psi\right) + \cdots$$

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Improvement: modify lattice action and operators to tune away $O(a)$ term. Can use axial Ward identities as conditions.

Auxiliary field

$$\mathcal{O}_\Gamma(z) = \bar{\psi}(z\mathbf{n})\Gamma U(z\mathbf{n}, 0)\psi(0)$$

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Introduce field ζ with $\langle \zeta(z\mathbf{n})\bar{\zeta}(0)\rangle_\zeta = U(z\mathbf{n}, 0)$.

$$\mathcal{O}_\Gamma(z) = \left\langle \bar{\phi}(z\mathbf{n})\Gamma\phi(0)\right\rangle_\zeta, \quad \phi \equiv \bar{\zeta}\psi$$

ϕ is a local operator in extended theory QCD+ζ.

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Symanzik expansion:

$$\phi^{(0)} = \phi + c\,\not{\mathbf{n}}\phi, \qquad \phi^{(1)} = c'(\mathbf{n}\cdot\partial)\phi + \cdots$$

Existence of preferred direction **n** leads to:

1. Chiral symmetry breaking allows mixing in $\phi^{(0)}$.
2. $O(a)$ contributions exist without breaking chiral symmetry.

Continuum limit study

Three $N_f = 2 + 1 + 1$ twisted mass Wilson ensembles from ETMC.

Name	a (fm)	size	m_π (MeV)	p_z (GeV)	N_{samp}
A60	0.0934	$24^3 \times 48$	365	1.66	40320
B55	0.0820	$32^3 \times 64$	373	1.89	58528
D45	0.0644	$32^3 \times 64$	371	1.80	40288

Study isovector unpolarized and helicity PDFs,

$$q(x) = u(x) - d(x), \qquad \Delta q(x) = \Delta u(x) - \Delta d(x).$$

Negative x is antiquark region,

$$\bar{q}(x) = -q(-x), \qquad \Delta\bar{q}(x) = \Delta q(-x).$$

Continuum limit study: renormalization

Two approaches used to obtain $\overline{MS}$ position-space matrix elements.

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Whole operator

$$h^{\text{bare}}(z, p_z, a) \xrightarrow{\text{NPR}} h^{\text{RI}'\text{-MOM}}(z, p_z, \mu) \xrightarrow[C(z)]{O(\alpha_s)} h^{\overline{\text{MS}}}(z, p_z, \mu)$$

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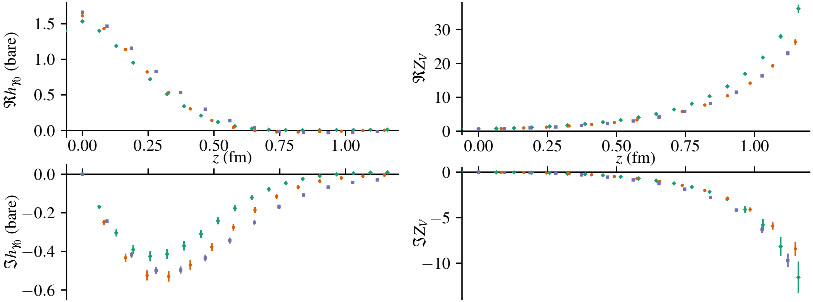
Final steps are the same in both cases:

$$h^{\overline{\text{MS}}}(z, p_z, \mu) \xrightarrow{O(\alpha_s)} h^{\text{M}\overline{\text{MS}}}(z, p_z, \mu) \xrightarrow{\text{F.T.}} \tilde{q}^{\text{M}\overline{\text{MS}}}(x, p_z, \mu) \xrightarrow[\text{match}]{O(\alpha_s)} q^{\overline{\text{MS}}}(x, \mu).$$

The $\text{M}\overline{\text{MS}}$ scheme removes a small-z divergence and provides a matching that conserves charge.

Bare matrix elements and renormalization factors



coarse (blue), medium (orange), fine (green)

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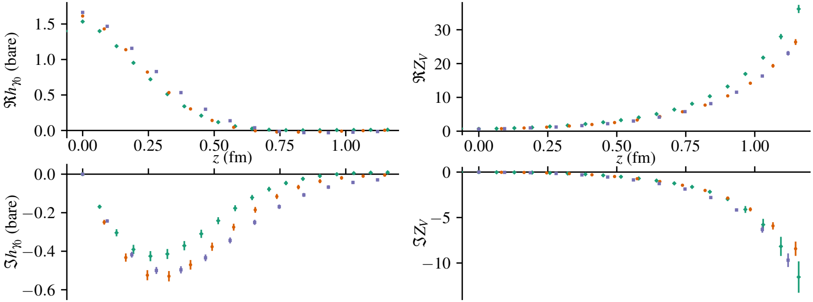
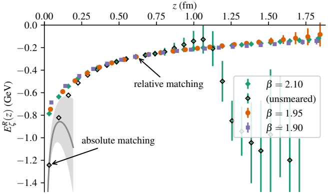


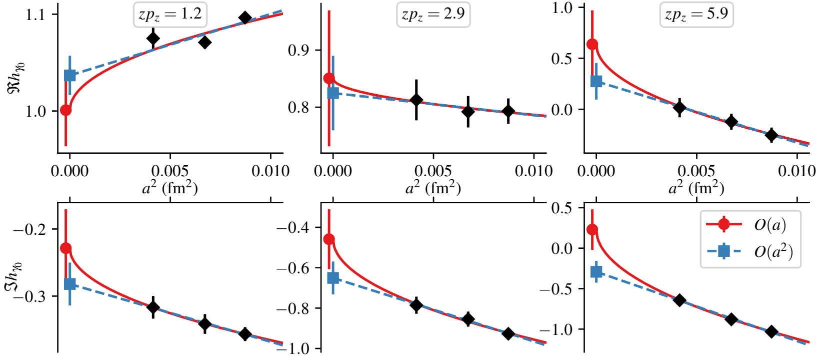
coarse (blue), medium (orange), fine (green)

Auxiliary field: $O^R(z) = Z_\phi^2(1 - r_{\text{mix}}^2)e^{-m|z|}O(z)$.

	am	$Z_\phi^2(1 - r_{\text{mix}}^2)$
coarse	−0.392(1)(57)	0.986(25)
medium	−0.373(1)(50)	0.908(20)
fine	−0.305(1)(37)	0.907(11)

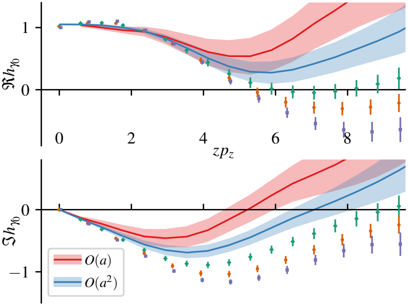


Continuum extrapolation (whole operator approach)

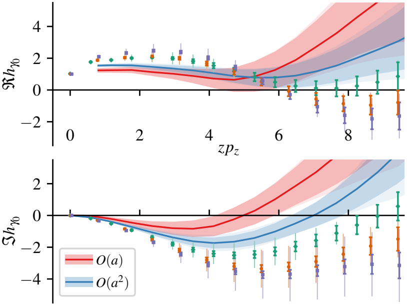


Hard to distinguish $O(a)$ from $O(a^2)$ in data.

Continuum limit (position space)

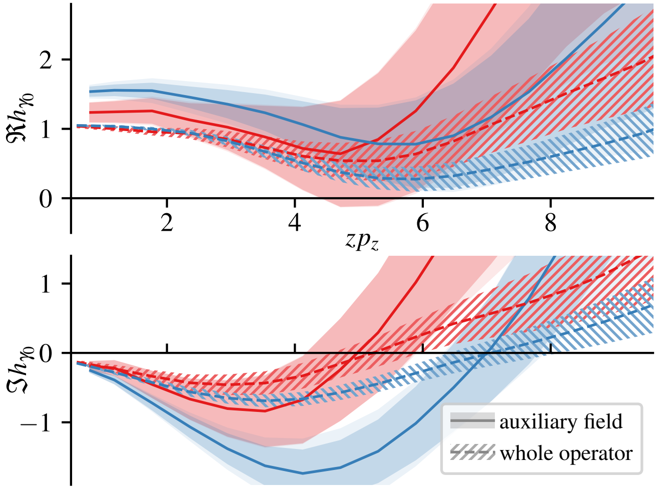


whole operator



auxiliary field

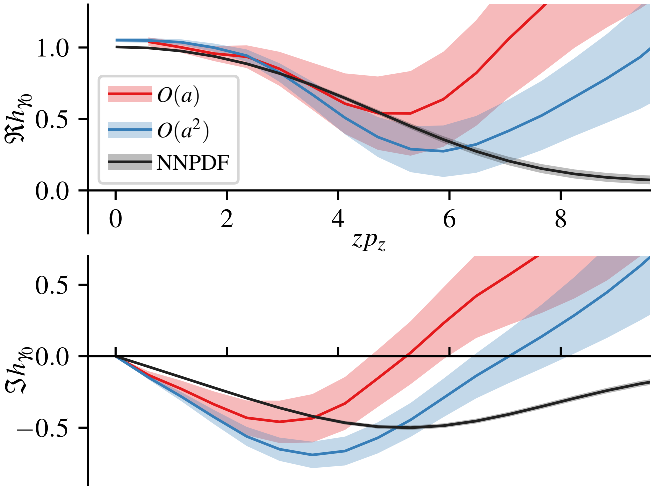
Continuum limit: whole operator vs. auxiliary field



red: $O(a)$, blue: $O(a^2)$

Focus on whole operator approach for PDF study.

Comparison with phenomenology (position space)



Recall $m_\pi \approx 370$ MeV.

Quasi-PDF

Defined as Fourier transform of $h(z)$:

$$\tilde{q}(x, p_z) \equiv \frac{p_z}{2\pi} \int dz\, e^{-ixp_z z} h(z, p_z).$$

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We only have finite usable data.
Truncated discrete FT (**DFT**):

$$\tilde{q}^{\mathrm{DFT}}(x, p_z) \equiv \frac{p_z}{2\pi} a \sum_{z/a=-z_{\max}/a}^{z_{\max}/a} e^{-ixp_z z} h(z, p_z)$$

$$\xrightarrow{a\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} dx' \, \frac{\sin((x - x')z_{\max} p_z)}{(x - x')} \tilde{q}(x', p_z).$$

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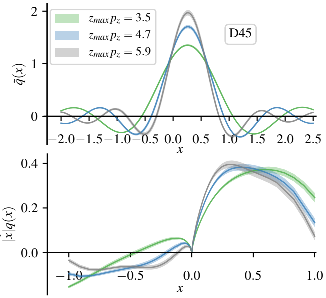

Reconstruction techniques 1

Backus-Gilbert (BG): write as inverse problem.

$$\Re h(z, p_z) = \int_0^\infty dx\, \cos(x p_z z) \tilde{q}_+(x, p_z)$$

$$\Im h(z, p_z) = \int_0^\infty dx\, \sin(x p_z z) \tilde{q}_-(x, p_z)$$

where $\tilde{q}_\pm = \tilde{q}(x) \pm \tilde{q}(-x)$.

G. Backus and F. Gilbert, Geophys. J. Int. **16**, 169 (1968)

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Assume the solution is linear in the data:

$$\tilde{q}_+^{\text{BG}}(x) \equiv \sum_{z/a=0}^{z_{\max}/a} \mathbf{a}_+(x, z) \Re h(z) = \int_0^\infty dx'\, \Delta_+(x, x') \tilde{q}_+(x').$$

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Determine $\mathbf{a}_+$ by minimizing width of resolution function

$$\Delta_+(x, x') \equiv \sum_{z/a=0}^{z_{\max}/a} \mathbf{a}_+(x, z) \cos(x' p_z z).$$

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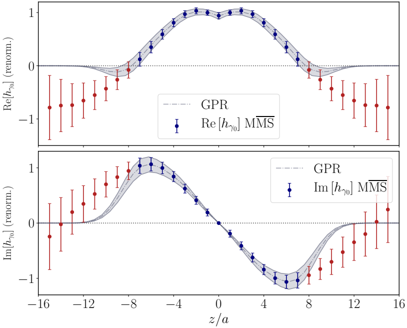
Reconstruction techniques 2

Bayes-Gauss-Fourier Transform (BGFT)

Use Gaussian process regression (Bayesian inference) to reconstruct continuous position-space matrix element.

$$h(z), z/a \in \{0, \pm 1, \pm 2, \ldots, \pm z_{\text{max}}/a\} \longrightarrow h^{\text{GPR}}(z), z \in \mathbb{R}$$

Imposes smoothness and decay to zero at large $|z|$.



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Fourier transform computable in closed form.

$$\tilde{q}^{\text{BGFT}}(x, p_z) \equiv \frac{p_z}{2\pi} \int dz\, e^{-ixp_z z} h^{\text{GPR}}(z, p_z)$$

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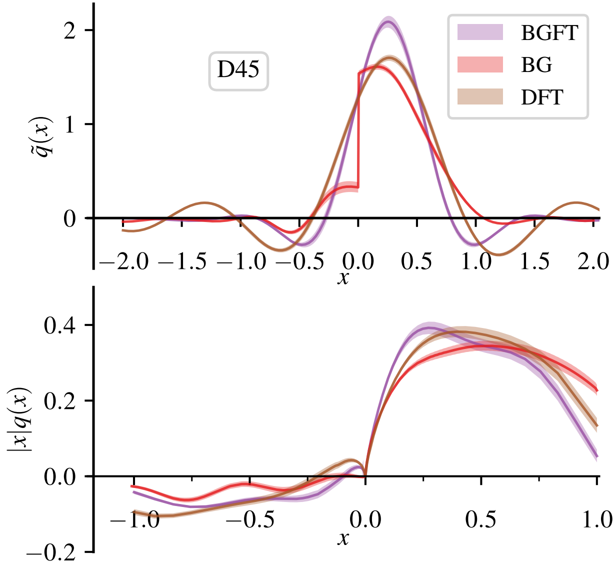
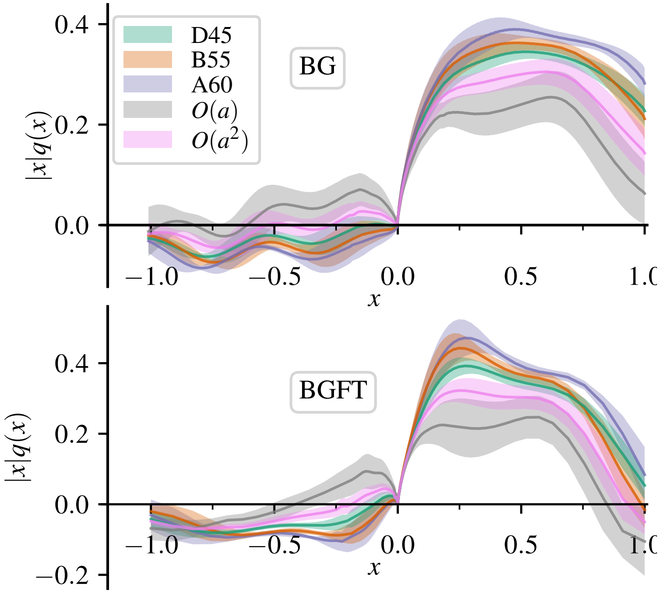
Reconstruction techniques: comparison



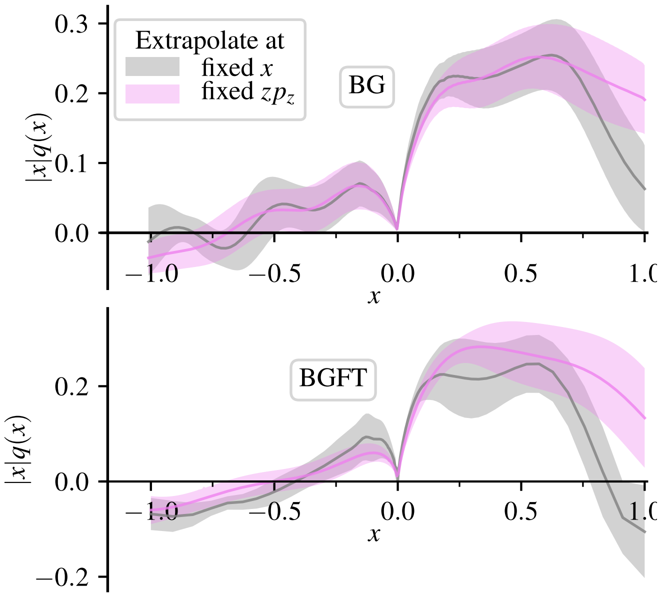
Focus on BGFT and BG methods for continuum extrapolation.

Continuum limit (PDF)



Independent extrapolation at each x.

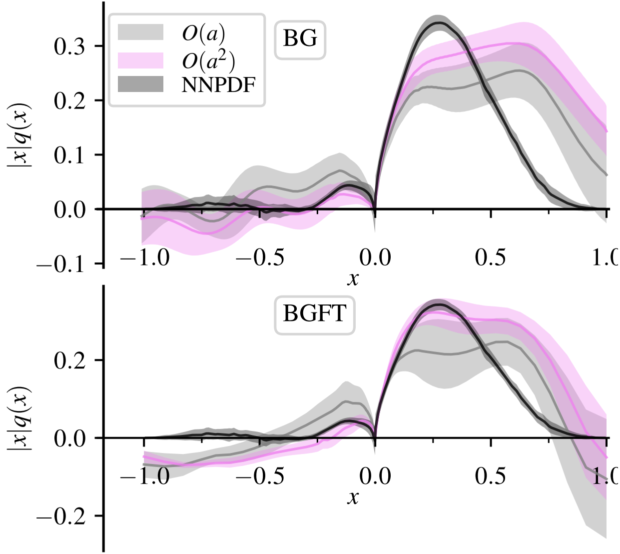
Order of operations



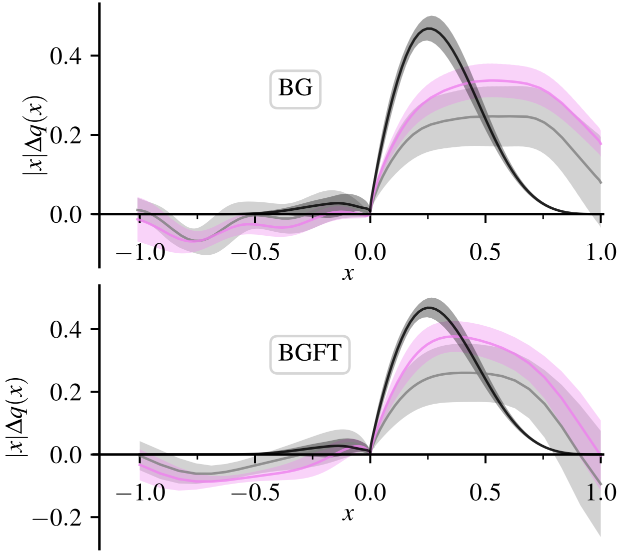
$O(a)$ extrapolations.

Extrapolate $a \to 0$ for $h(z)$ or $q(x)$.

Comparison with phenomenology



Helicity


BG
0.4
0.2
0.0
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−0.5
0.0
0.5
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x
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−0.5
0.0
0.5
1.0
x

Outlook

- Discretization effects are important. Can we apply $O(a)$ improvement?
- Systematics may lurk in renormalization / matching.
  - Higher order perturbative calculations needed.
  - Can the discrepancies between whole operator and auxiliary field methods be understood?

Excited-state effects