Linear divergence under lattice regularization

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Based on K. Zhang, et.al, χQCD, 2012.05448 and further on-going works

Large momentum effective theory

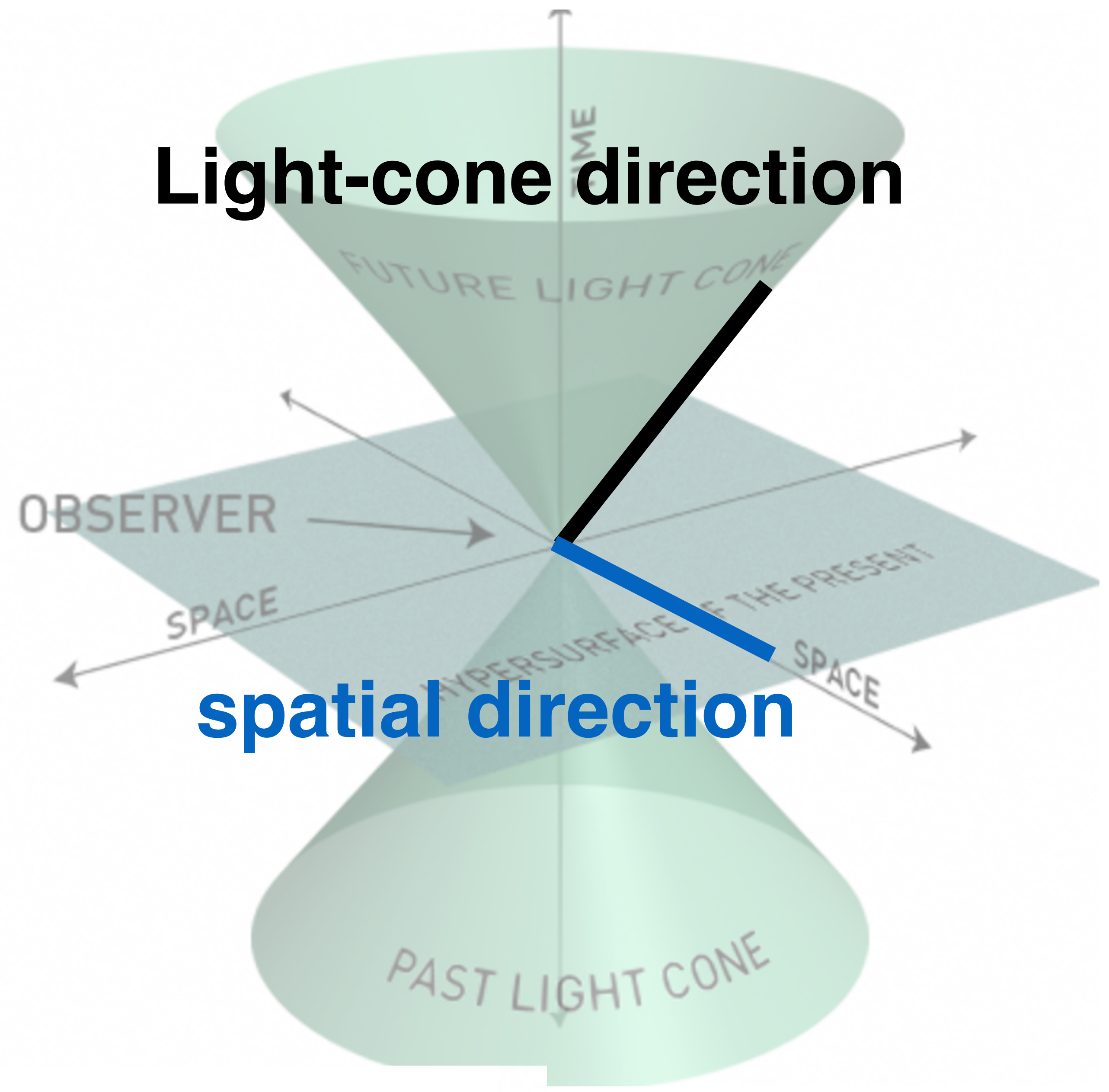
quasi-PDF

The light-cone PDF is defined by

$$q(x,\mu^2) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^- P^+} \langle P|\bar{\psi}(\xi^-)\gamma^+ \times \exp\left(-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)\right)\psi(0)|P\rangle$$

and can be accessed by,

$$q(x,\mu) = \int_{-\infty}^{\infty} \frac{dy}{|y|} \left\{\delta\left(1-\frac{x}{y}\right) - \frac{\alpha_s C_F}{2\pi} C_1\left(\frac{x}{y}, \frac{yP_z}{\mu}, \frac{yP_z}{p_z^R}, \frac{\mu_R^2}{p_z^{R2}}\right)\right\} \int_{-\infty}^{\infty} \frac{e^{iyP_z z}}{4\pi} \langle P|\bar{\psi}(z)\gamma_t U_z(z,0)\psi(0)|P\rangle^R_{p^2=\mu_R^2, p_z=p_z^R} + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{P_z^2}, \alpha_s^2\right),$$



$$\langle P|\bar{\psi}(z)\gamma_t U_z(z,0)\psi(0)|P\rangle^R_{p^2=\mu_R^2, p_z=p_z^R}$$

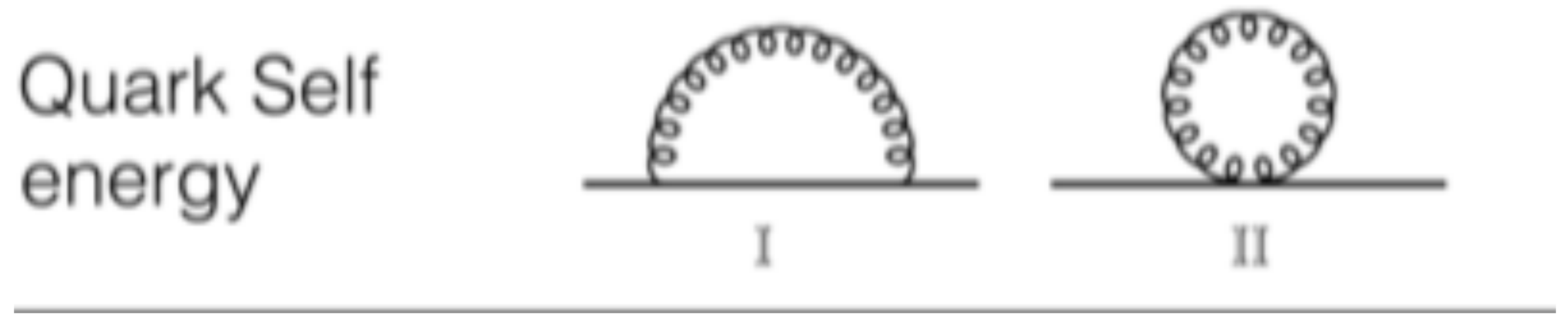
with the lattice calculation of the ***proper renormalized quasi-PDF****,*

$$\langle P\,|\,\mathcal{O}_t(z)\,|\,P\rangle^R_{p^2=\mu_R^2, p_z=p_z^R}, \quad \mathcal{O}_t = \bar{\psi}(0)\gamma_t U_z(0,z)\psi(z)$$

But

The lattice regularization is highly non-trivial

- The lattice regularization is sensitive to the fermion and gauge actions we used.
- The corresponding Feynman rules are sensitive;
- The quantum corrections are also sensitive.
- The conclusion using the dimensional/cut-off regularization, should be verified under the lattice regularization with several fermion and gauge actions.



The quark self energy with different discretized actions are:

$$Z_Q^{RI}(p^2) = 1 + \frac{g^2 C_F}{16\pi^2}[(1-\xi)\log(a^2p^2) + B_Q + 4.79\xi] + O(a^2p^2) + O(g^4)$$

Gauge actions

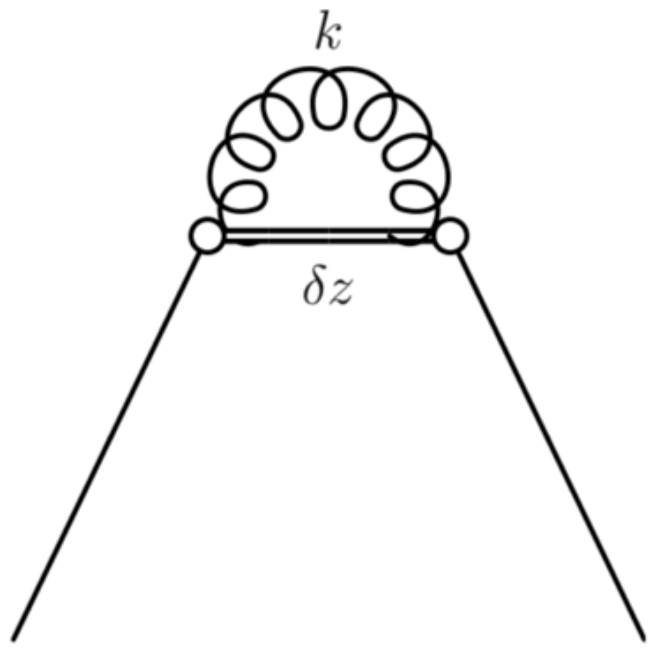
Fermion actions

	B_Q		
	Wilson	Iwasaki	Iwasaki^{HYP}
Wilson	11.85	3.32	-4.22
overlap	-21.50	-13.58	-7.56

Tadpole diagrams

Under Lattice regularization

$$O_{\delta z}^{(2)\mu\nu,AB}(p,q,k) = -g^2\{T^A,T^B\}\gamma_3\delta^{\mu 3}\delta^{\nu 3}\delta(p-q)e^{-ip_3\delta z}\frac{2\sin^2(\frac{k_3}{2}\delta z)}{(\frac{2}{a}\sin\frac{k_3 a}{2})^2}$$



$$\int_{-\pi}^{\pi}\frac{dp^4}{(2\pi)^4}S_g(p)_{\mu\mu}\frac{\sin^2\left(\frac{z}{a}\frac{p_\mu}{2}\right)}{\sin^2(p_\mu/2)} = \int_{-\pi}^{\pi}\frac{dp^3}{(2\pi)^3}S_g(\overline{p})_{\mu\mu}\cdot\int_{-\pi}^{\pi}\frac{dp_\mu}{(2\pi)}\frac{\sin^2\left(\frac{z}{a}\frac{p_\mu}{2}\right)}{p_\mu^2/4}+\mathcal{O}(a^0,\,\log a)$$
$$= \int_{-\pi}^{\pi}\frac{dp^3}{(2\pi)^3}S_g(\overline{p})_{\mu\mu}\cdot\left(-\frac{2}{\pi^2}+\frac{|z|}{a}\right)+\mathcal{O}(a^0,\,\log a)\,,$$

Action	e_2
Wilson	-19.9548
TL Symanzik	-17.2937
Iwasaki	-12.9781

M. Constantinou, et.al., 1705.11193

$$Z_\Gamma(z) = 1+\frac{\alpha_s C_F}{4\pi}(e_2\frac{|\delta z|}{a}+\ldots\ldots)$$

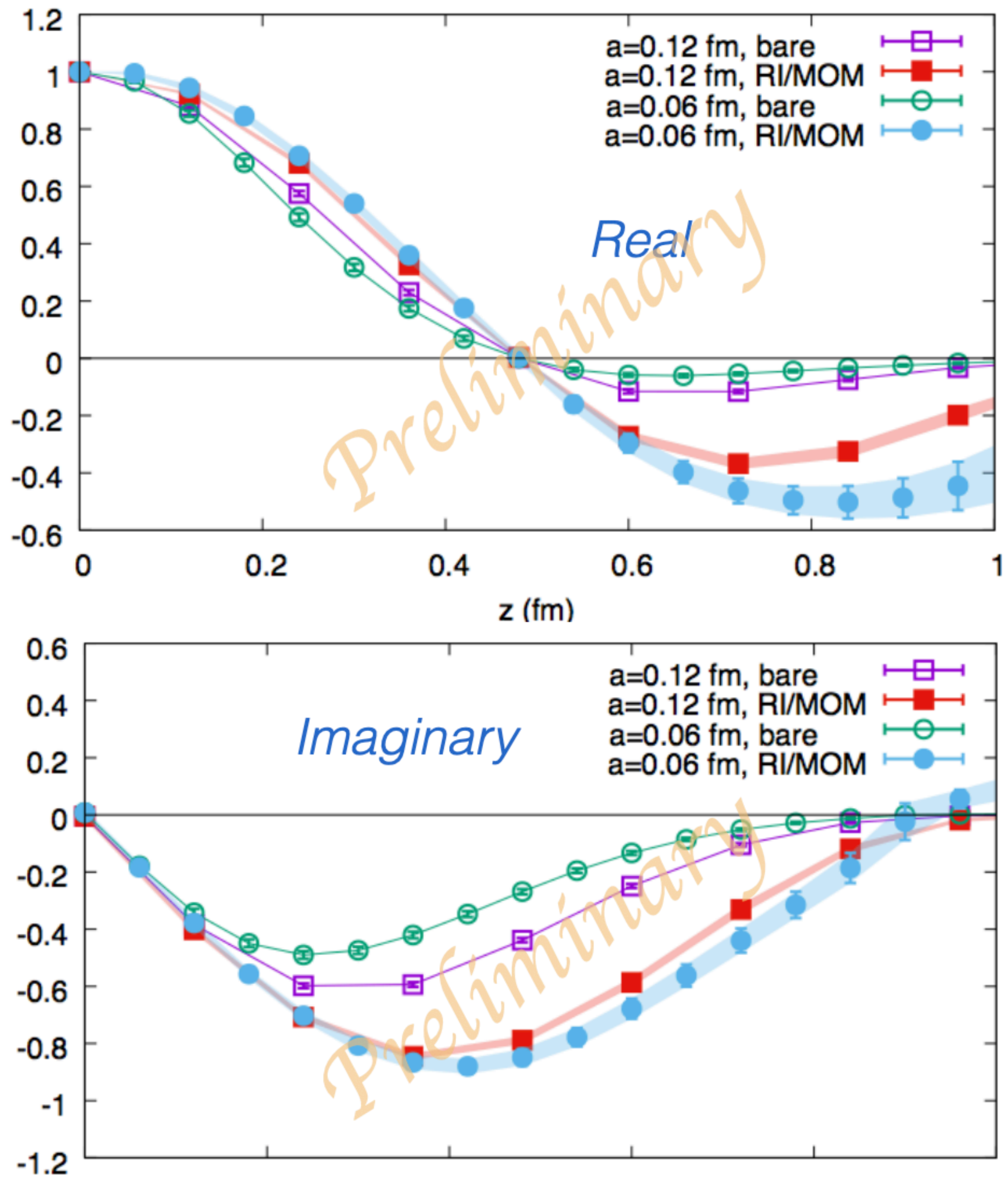
But the cutoff case is

$$-2\pi^2 = -19.7392\ldots\ldots$$

- The slide I used three years ago at USQCD all hands meeting 2018.
- The linear divergence in DA seems to be cancelled while the uncertainty at a=0.06 fm is still large.
- Whether we can have a more accurate check?

Linear divergence cancellation

Example II



- *The **RI/MOM renormalized** and normalized quasi-PDA at a=0.06/0.12 fm:*

$$\langle \eta_s(P_z = 1.3\mathrm{GeV}) | \bar{\psi}(z)\gamma_z\gamma_5 U_z(z,0)\psi(0)|0\rangle$$

- *The renormalized results at a=0.12 fm and a=0.06 fm agree with each other well up to z~0.5 fm.*
- *The present statistics at a=0.06 fm is ~1/4 of that at a=0.12 fm. It will be improved to provide a stronger check.*

Outline

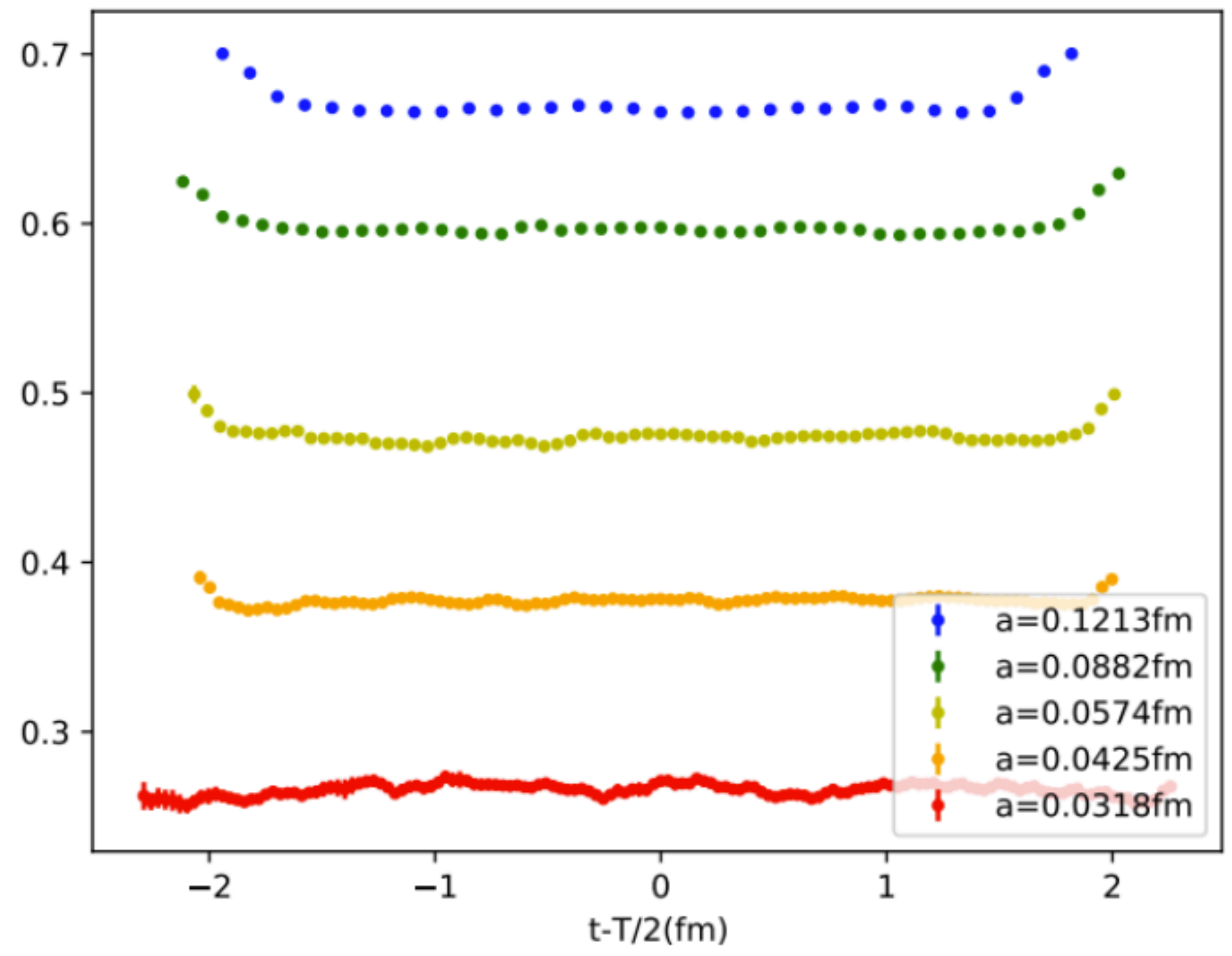
- **Hadron matrix elements**
- Vacuum matrix elements
- Quark matrix elements (RI/MOM)
- Discussion

Hadron matrix elements

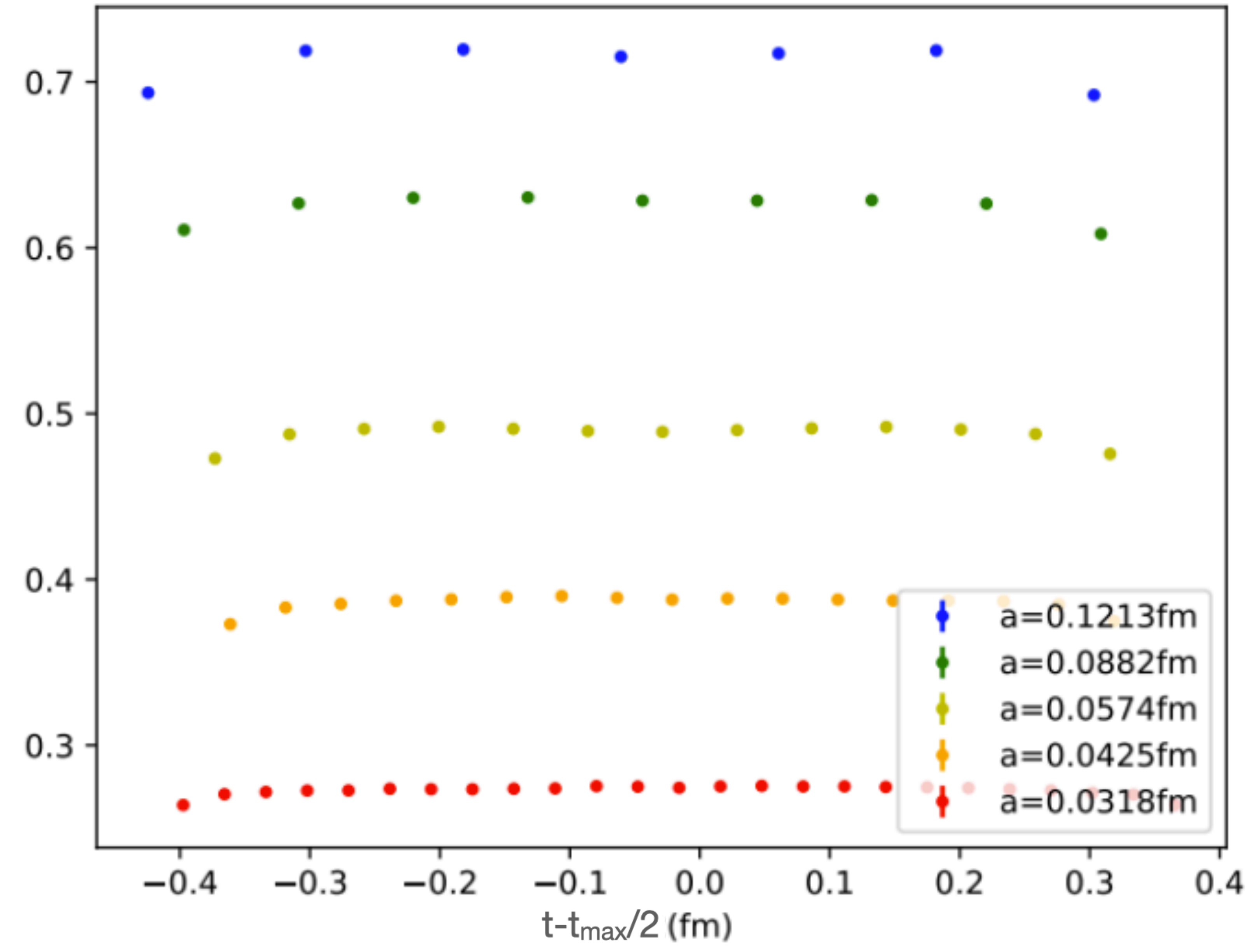
3pt (quasi-PDF matrix element)

$$R(t_2, t, z; H) \equiv \frac{\langle O_H(t_2) \sum_{\vec{x}} O_{\gamma_t}(z; (\vec{x}, t)) O_H^\dagger(0) \rangle}{\langle O_H(t_2) O_H^\dagger(0) \rangle}$$
$$= \langle H | O_{\gamma_t}(z) | H \rangle$$
$$+ \mathcal{O}(e^{-\delta m t}) + \mathcal{O}(e^{-\delta m (t_2 - t)}) + \mathcal{O}(e^{-\delta m t_2}),$$

- The linear divergence should be independent to the external state.
- And then one can switch to the rest frame for a more accurate check.
- The source-sink separation:
- Pion: T/2 (>3fm)
- Nucleon: ~0.72 fm



Pion

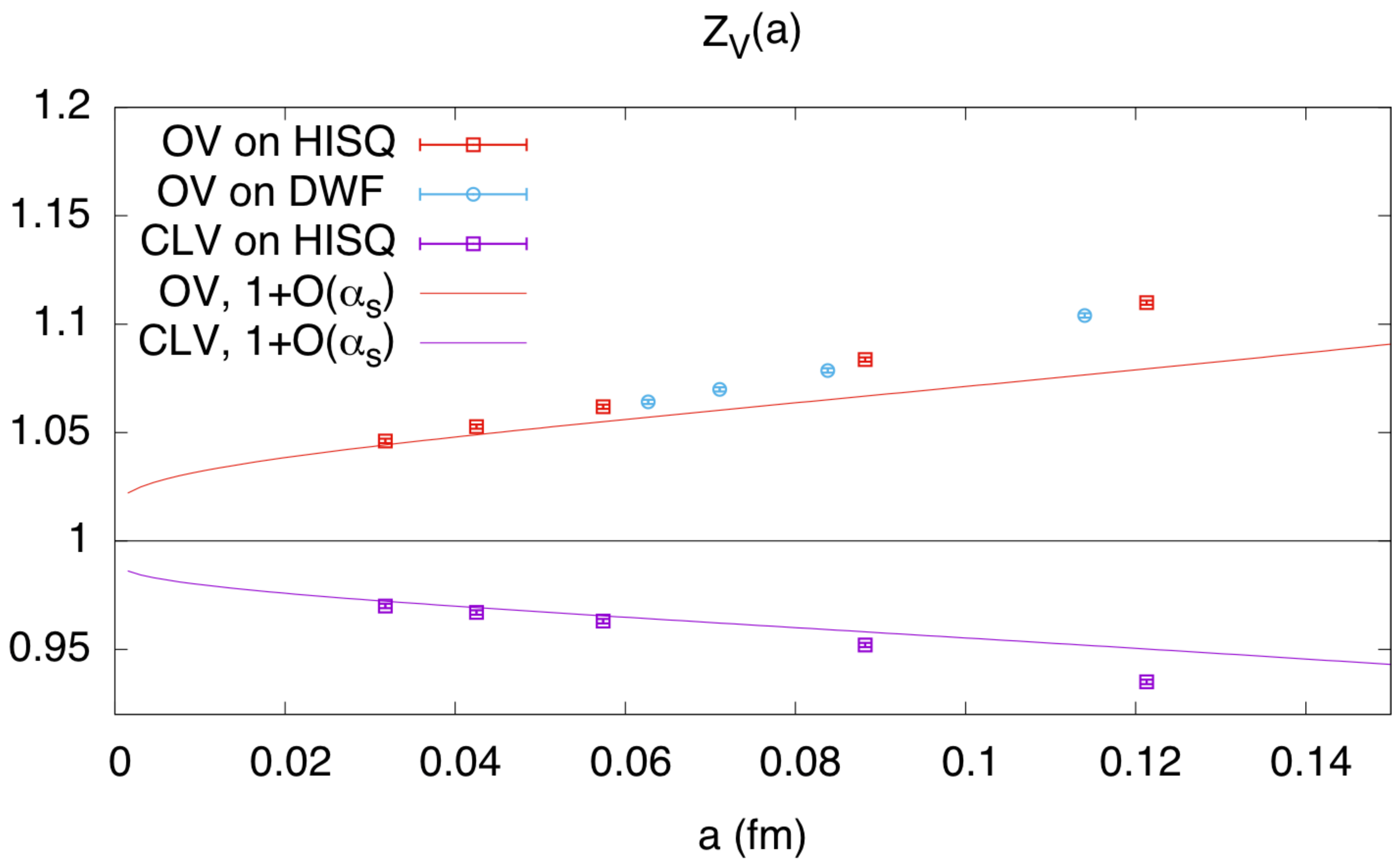


Nucleon

Hadron matrix elements

3pt (quasi-PDF matrix element)

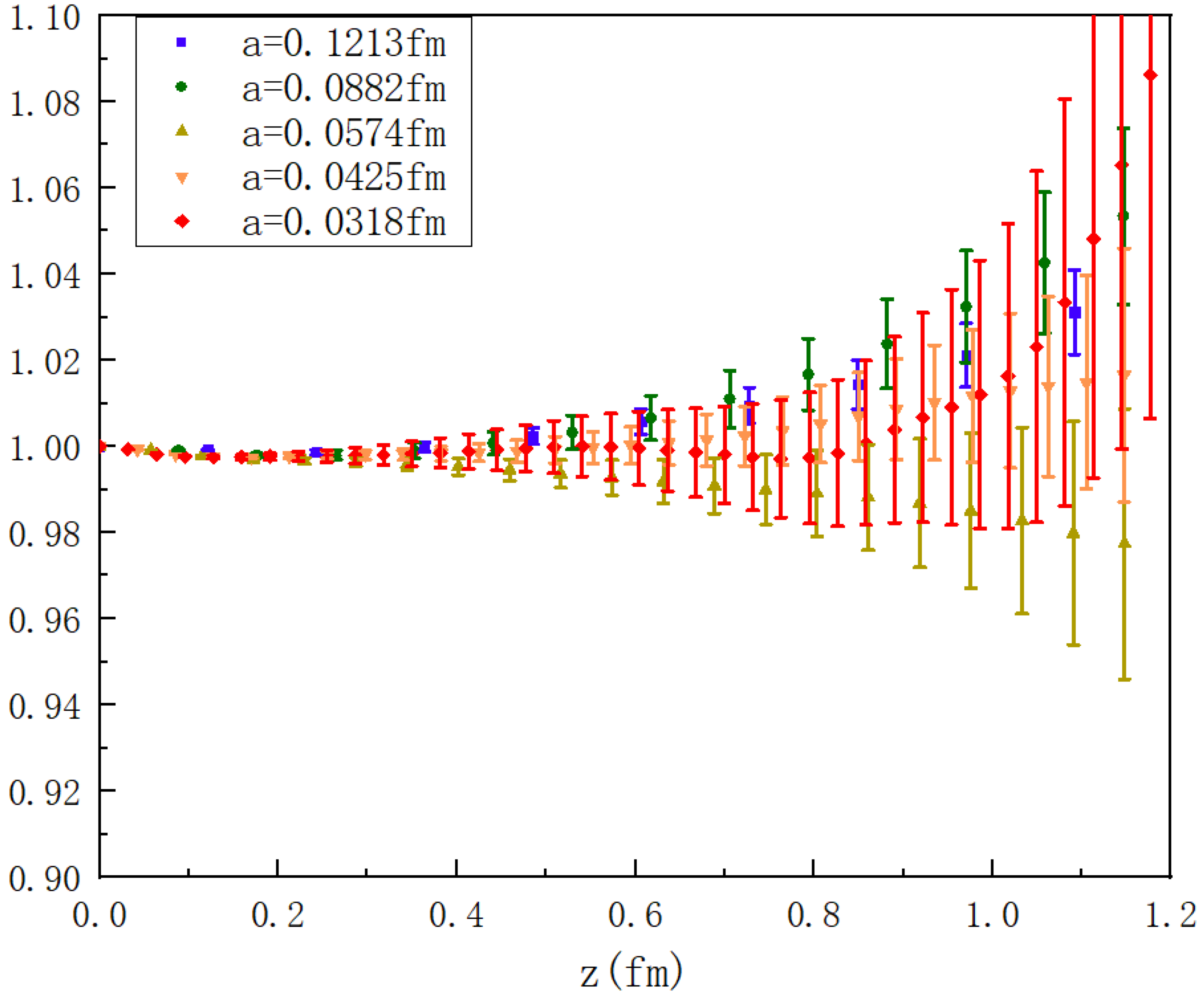
- Under the lattice regularization, a normalization is required to make the vector charge $\langle H|\mathcal{O}_t(0)|H\rangle$ to be integer.
- Such a normalization:
- A combination of the discretization error and also finite $\mathcal{O}(\alpha_s)$ correction.
- Not sensitive to the gauge action after the HYP smearing.
- Dominate by the $\mathcal{O}(\alpha_s)$ correction at small lattice spacing.



Hadron matrix elements

3pt (quasi-PDF matrix element)

$$\frac{\langle \pi | \mathcal{O}_t(z) | \pi \rangle_{clover}}{\langle \pi | \mathcal{O}_t(z) | \pi \rangle_{overlap}}$$



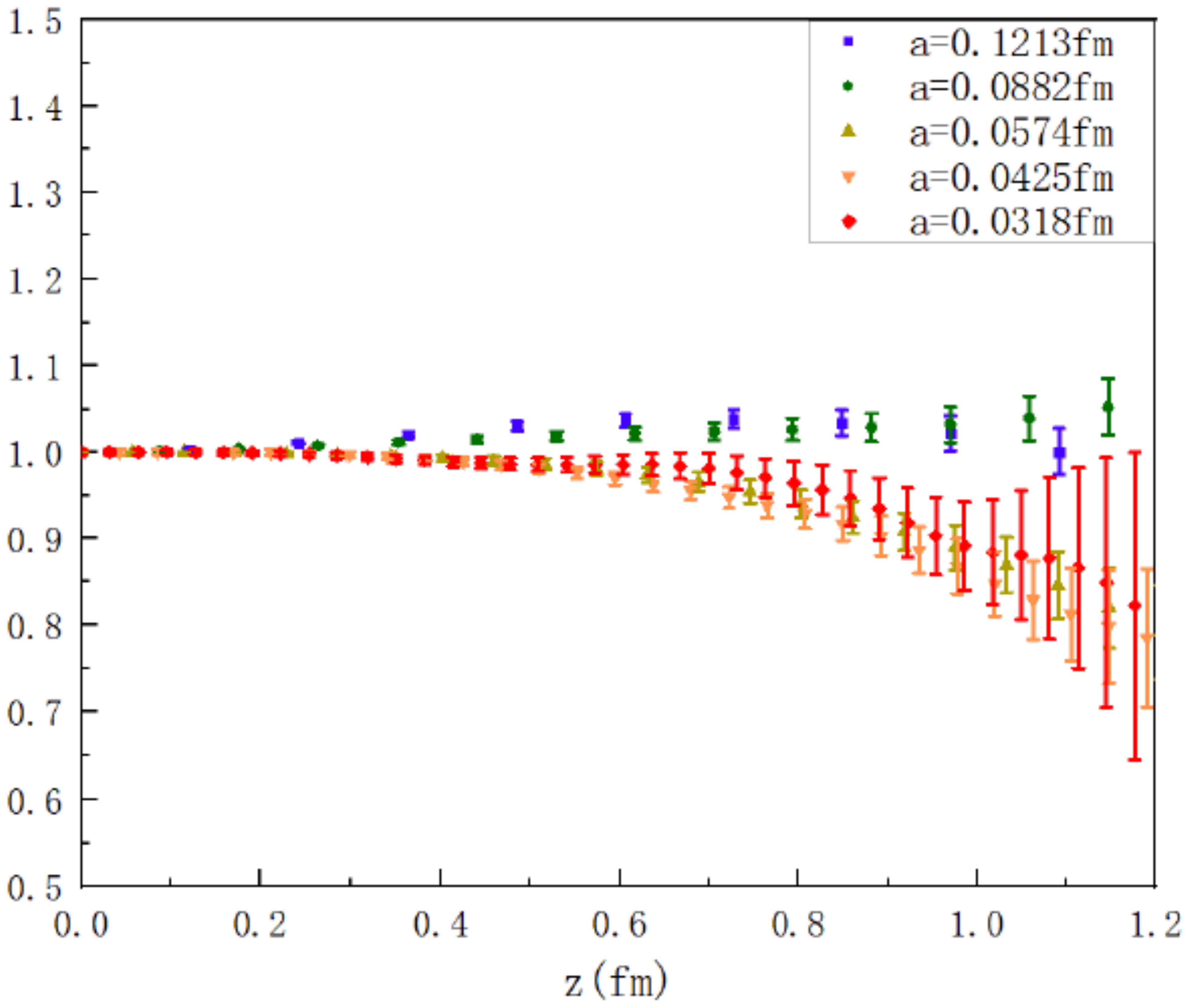
- We considered two fermion actions:
- Clover valence: widely used, cheap, with additive chiral symmetry breaking which is $\mathcal{O}(\alpha_s^2/a)$
- Overlap valence: expensive, perfect chiral symmetry
- HISQ sea: also chiral fermion, while can have the mixed action effect when the valence quark uses a different action.
- The pion matrix elements with two kinds of action are consistent with each other.

Hadron matrix elements

3pt (quasi-PDF matrix element)

$$\frac{\langle N | \mathcal{O}_t(z) | N \rangle}{\langle \pi | \mathcal{O}_t(z) | \pi \rangle}$$

- Ratio between the nucleon and pion matrix elements:
  - Some lattice spacing dependence at large z;
  - Seems to converge at small lattice spacing.
- Verified the argument on the external state independence of the linear divergence.



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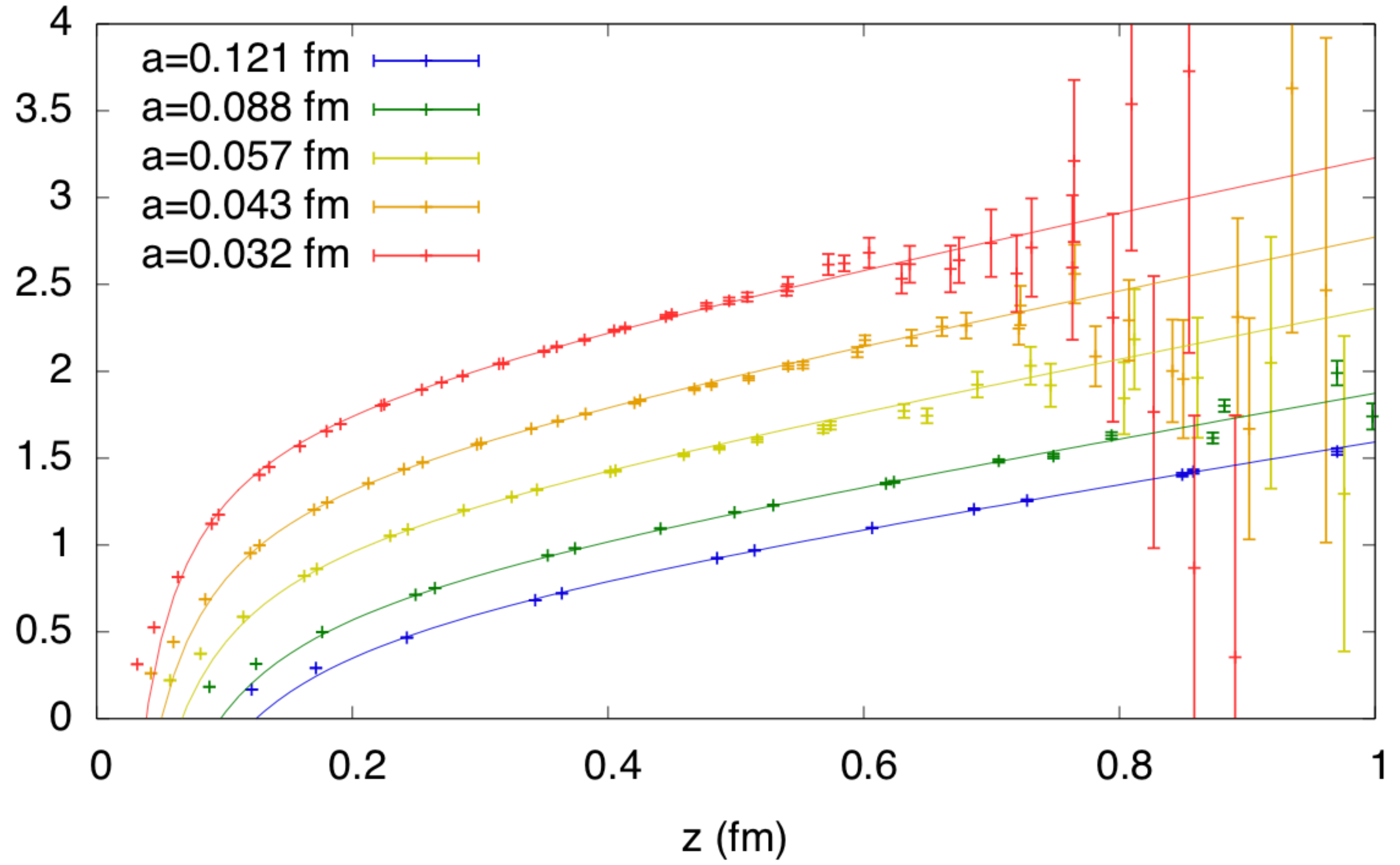
Vacuum matrix elements

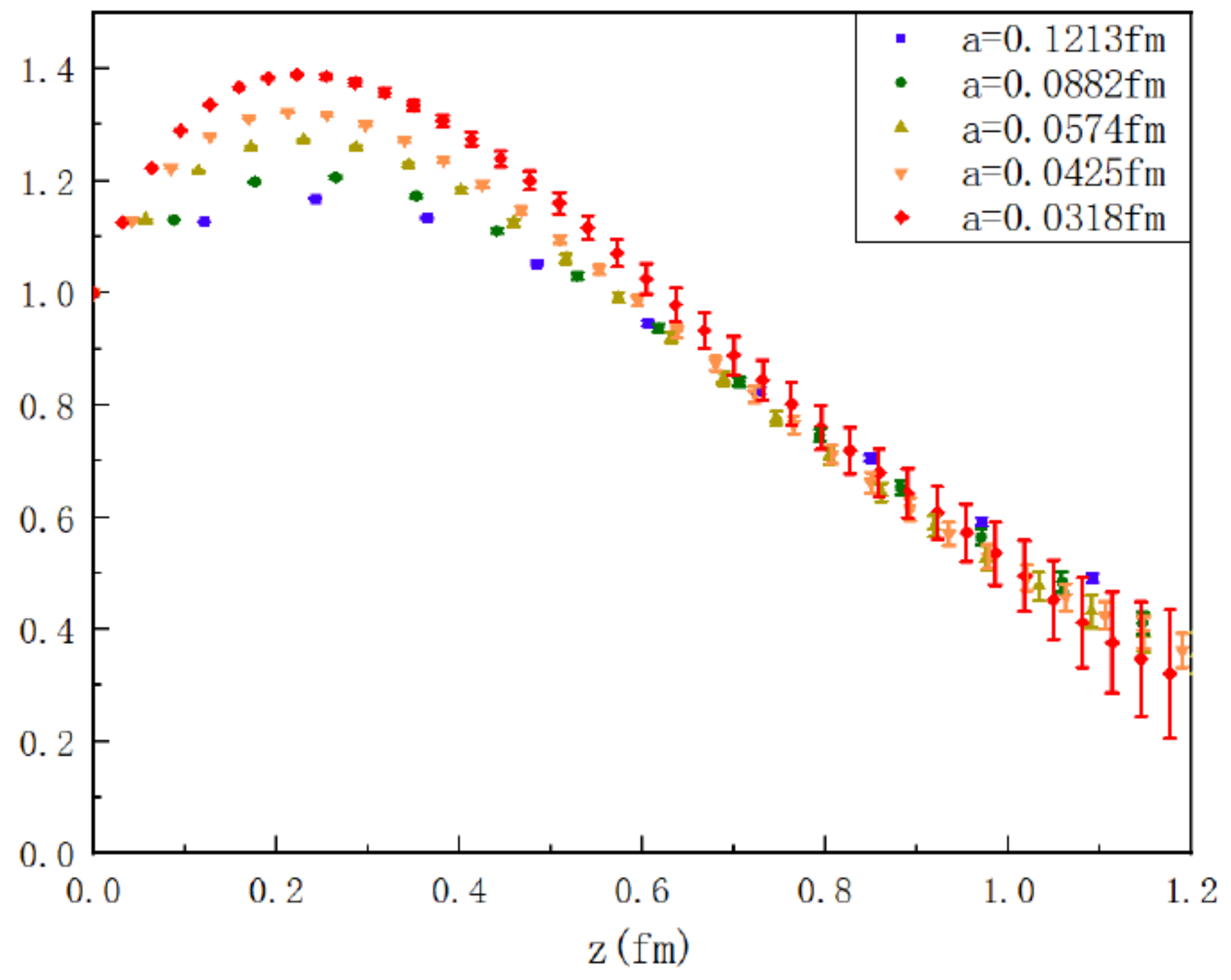
Wilson loop

- Based on the Wilson loop at different lattice spacings, we can have

$$\delta m \sim 0.2/a.$$

- We can renormalize the pion matrix elements with a renormalization factor $e^{\delta m\, z}$.
- The small z region still have obvious divergence.

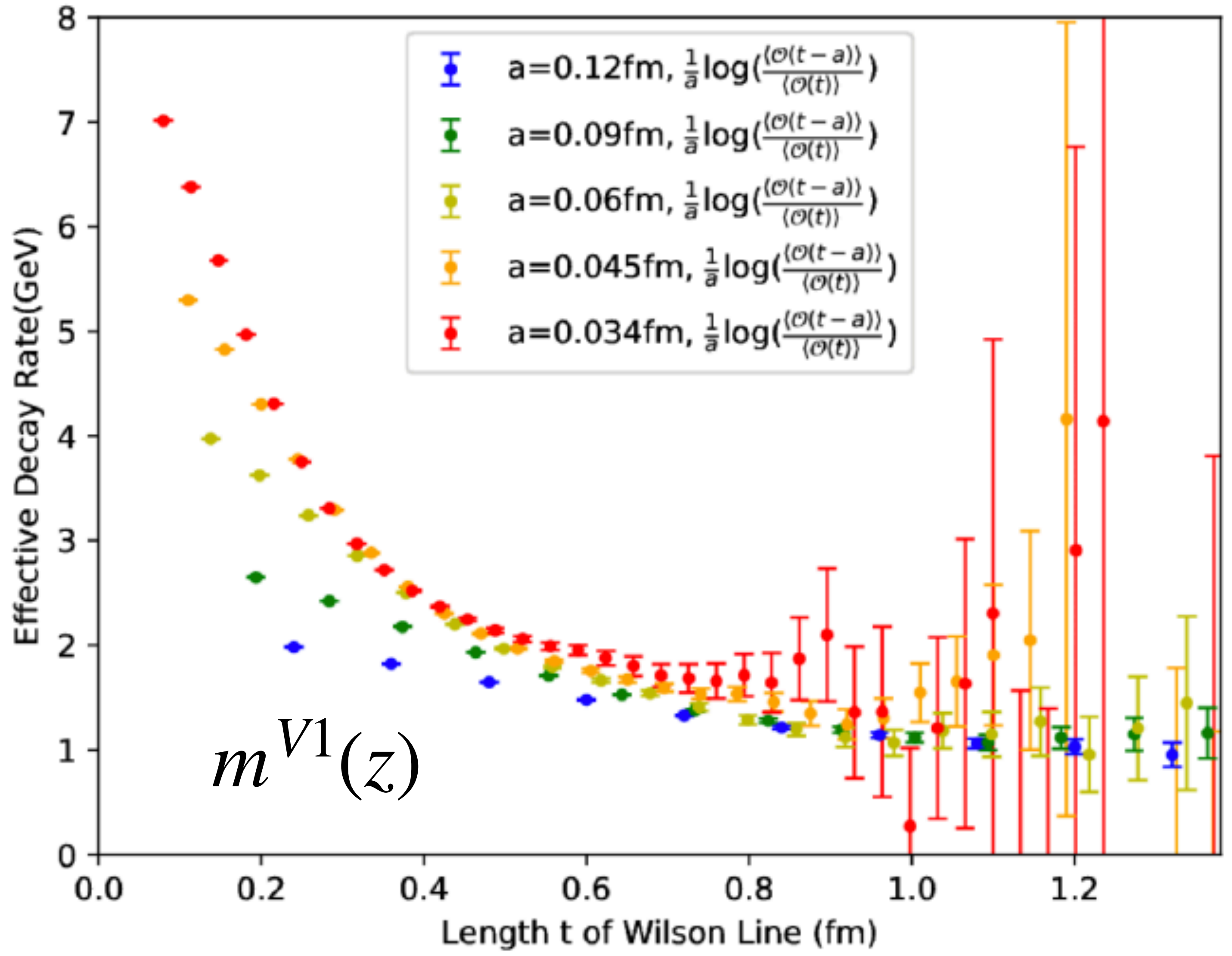




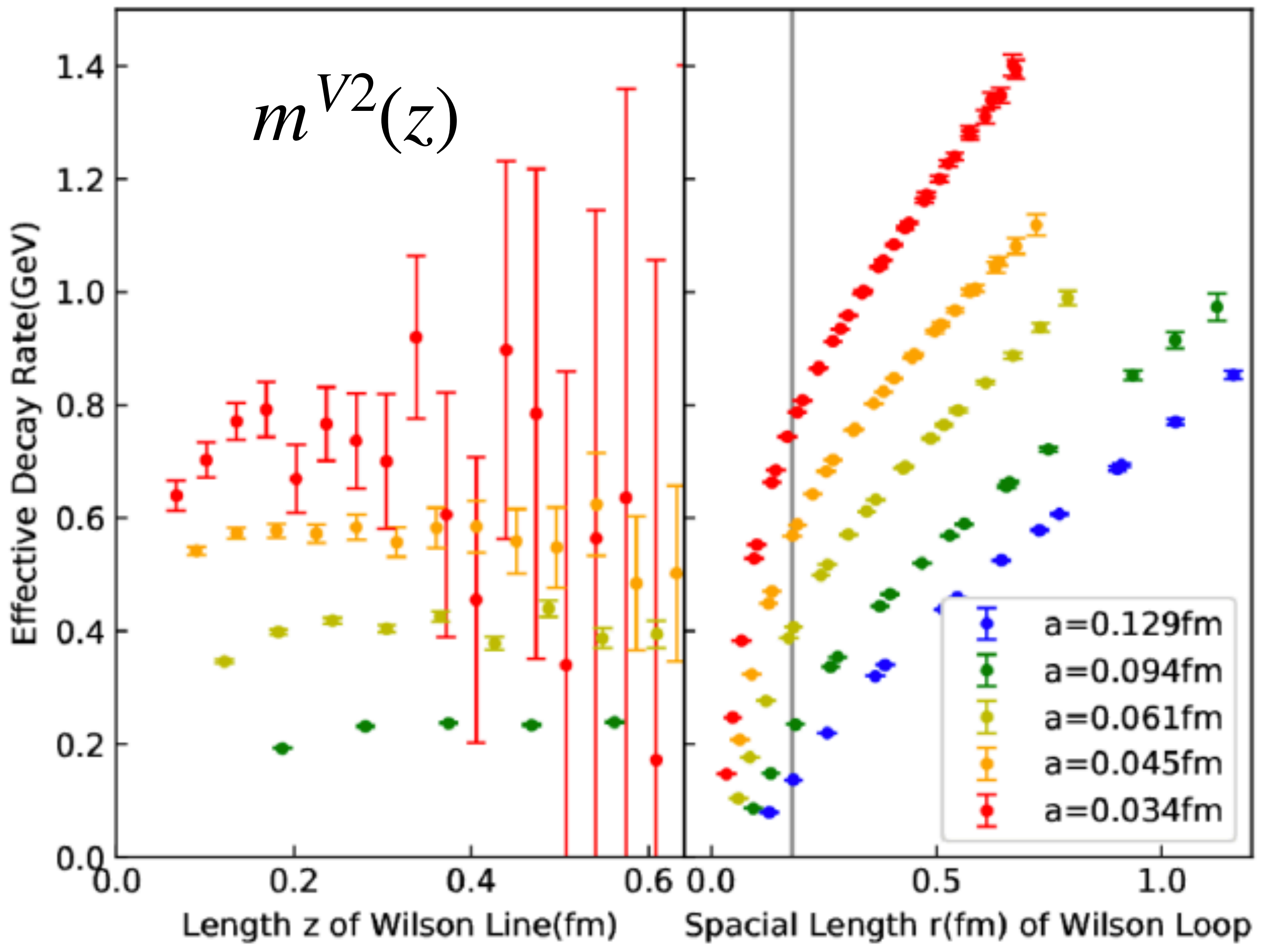
Vacuum matrix elements

quasi-PDF operator

- With the Clover fermion (the overlap case can be different).
- The effective decay rate of $\langle \mathcal{O}_t(t) \rangle$:
- $m^{V1}(z) = \frac{1}{a}\log(\frac{\langle \mathcal{O}_t(t-a) \rangle}{\langle \mathcal{O}_t(t) \rangle})$
- That of $\mathcal{O}_{sq}(z,t) = \langle \mathcal{O}_t(z,t)\mathcal{O}_t^\dagger(z,0) \rangle$:
- $m^{V2}(z) = \frac{1}{a}\log(\frac{\langle \mathcal{O}_{sq}(z-a,t_0) \rangle}{\langle \mathcal{O}_{sq}(z,t_0) \rangle})$
- That of Wilson loop $\langle U(r,t) \rangle$
- $m^{LP}(r) = \frac{1}{a}\log(\frac{\langle U(r,t-a) \rangle}{\langle U(r,t) \rangle})$
- As in the figure, we have $m^{V2}(z) \simeq m^{LP}(t_0)$



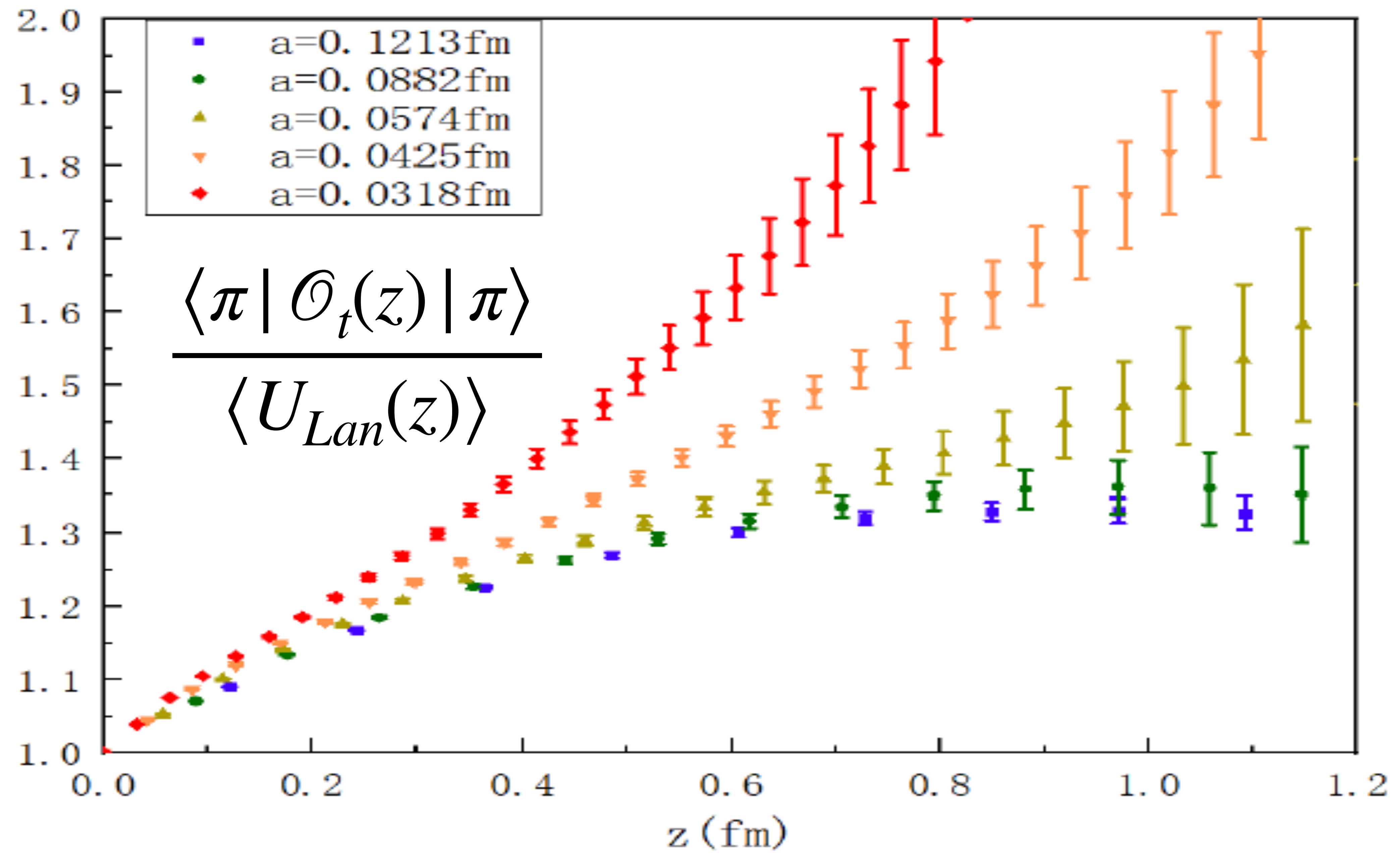
$m^{LP}(r)$

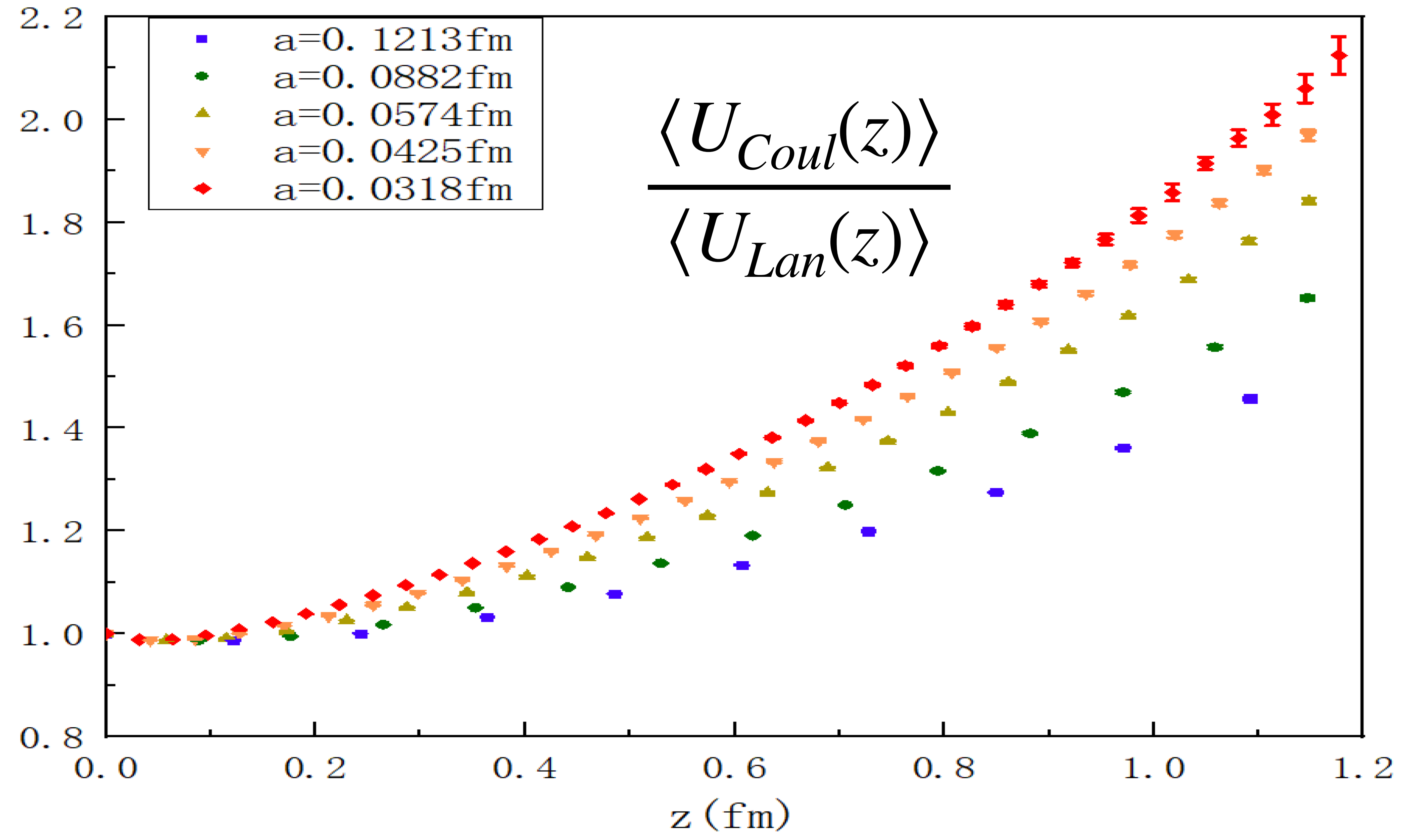


Vacuum matrix elements

Wilson line

- The vacuum expectation value can also be considered as a renormalization factor for the linear divergence.
- But the residual linear divergence is still obvious at all the z-region.
- An interesting observation is, the linear divergence from the Wilson link under two gauges seems to be the same in the continuum limit.





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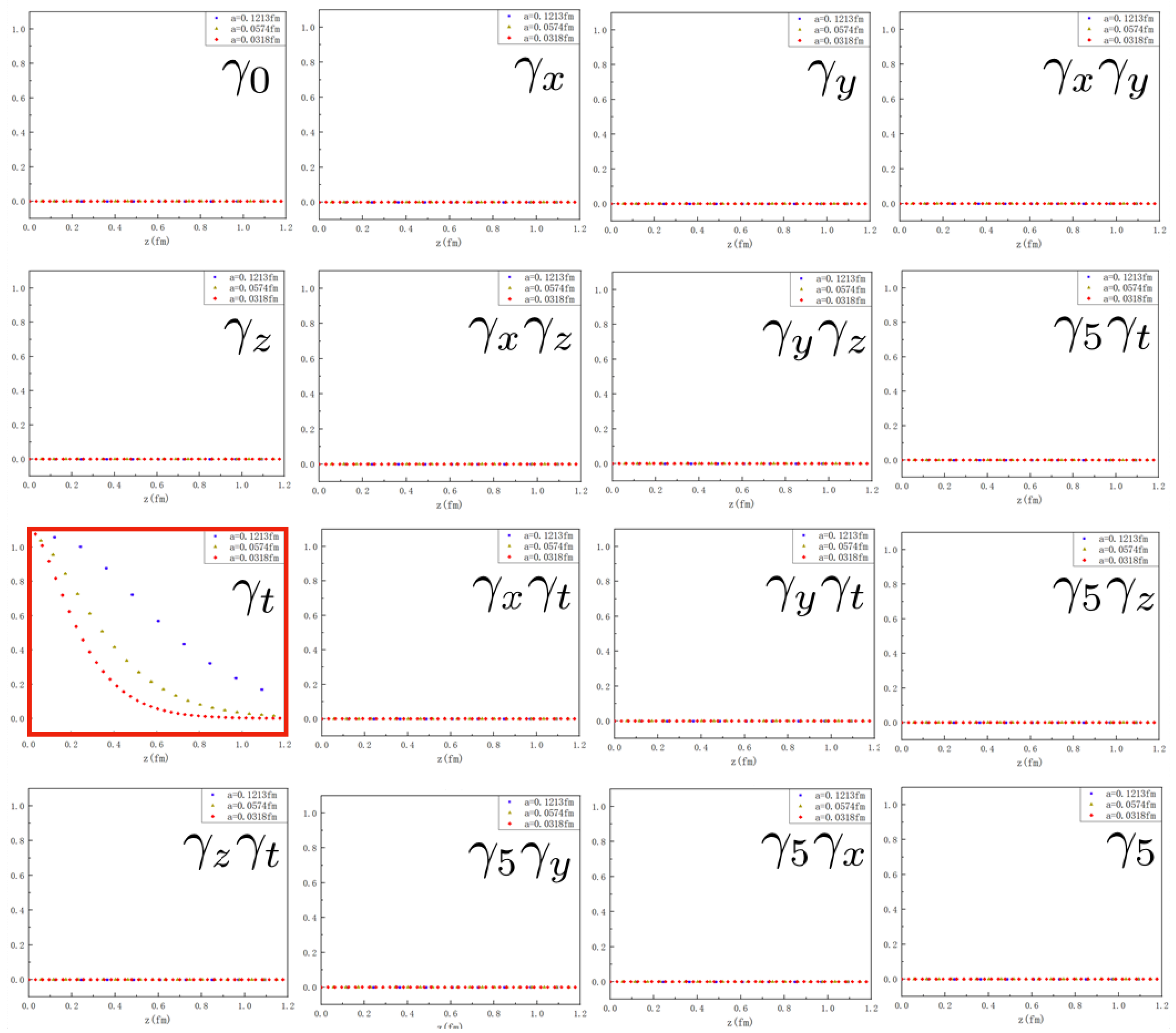
Quark matrix elements

Setup

$$\mathcal{M}_t^a(\Gamma, z) = \mathrm{Tr}[\Gamma \langle S(p)\rangle^{-1} \langle \sum_x S^\dagger(p,x)\gamma_t U_z(\vec{n}_z z) S(p, x+\vec{n}_z z)\rangle \langle S(p)\rangle^{-1}]/Z_q$$

$$\mathcal{M}_t^b(\Gamma, z) = \mathrm{Tr}[\gamma_t \langle S(p)\rangle^{-1} \langle \sum_x S^\dagger(p,x)\Gamma U_z(\vec{n}_z z) S(p, x+\vec{n}_z z)\rangle \langle S(p)\rangle^{-1}]/Z_q$$

- Momentum $(p_x, p_y, p_z, p_t) = (n, n, 0, 0)2\pi/L$;
- Wilson link along z direction;
- Z_q is defined from the vector current conservation and then we have $\mathcal{M}_t^{a,b}(\gamma_t, 0) = \langle \pi | \mathcal{O}_t | \pi \rangle$.
- Such a setup can avoid the mixing between gamma matrices completely.

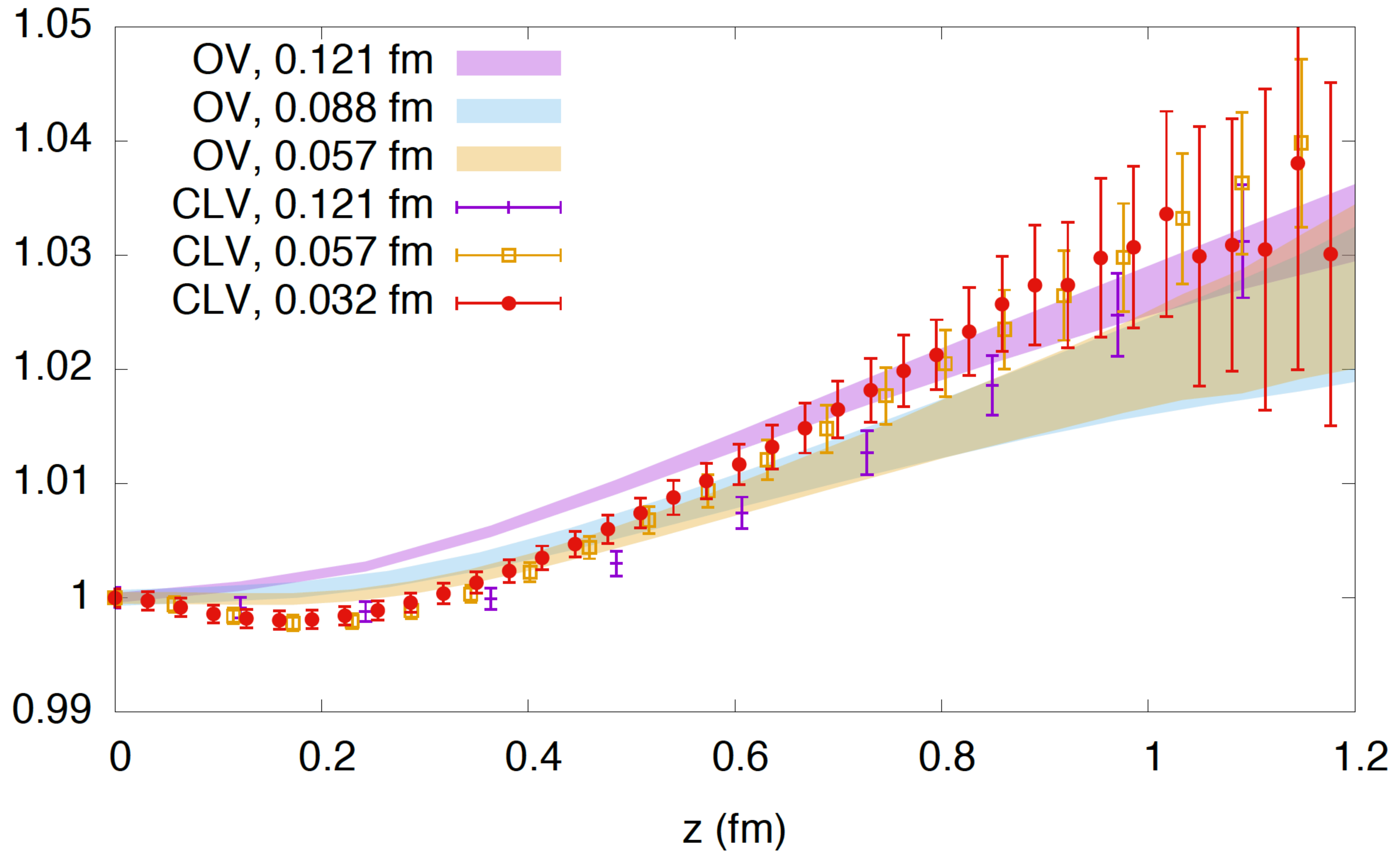


Quark matrix elements

Cancellation on UV divergence

- The ratios with different actions agree with each other in the continuum limit, with in sub-percentage uncertainty.
- It shows that the UV divergence including the linear one is independent of the external off-shell momentum.
- The other momenta (likes (0,0,0,0) or (7,7,0,0)) would be helpful to confirm the above argument.

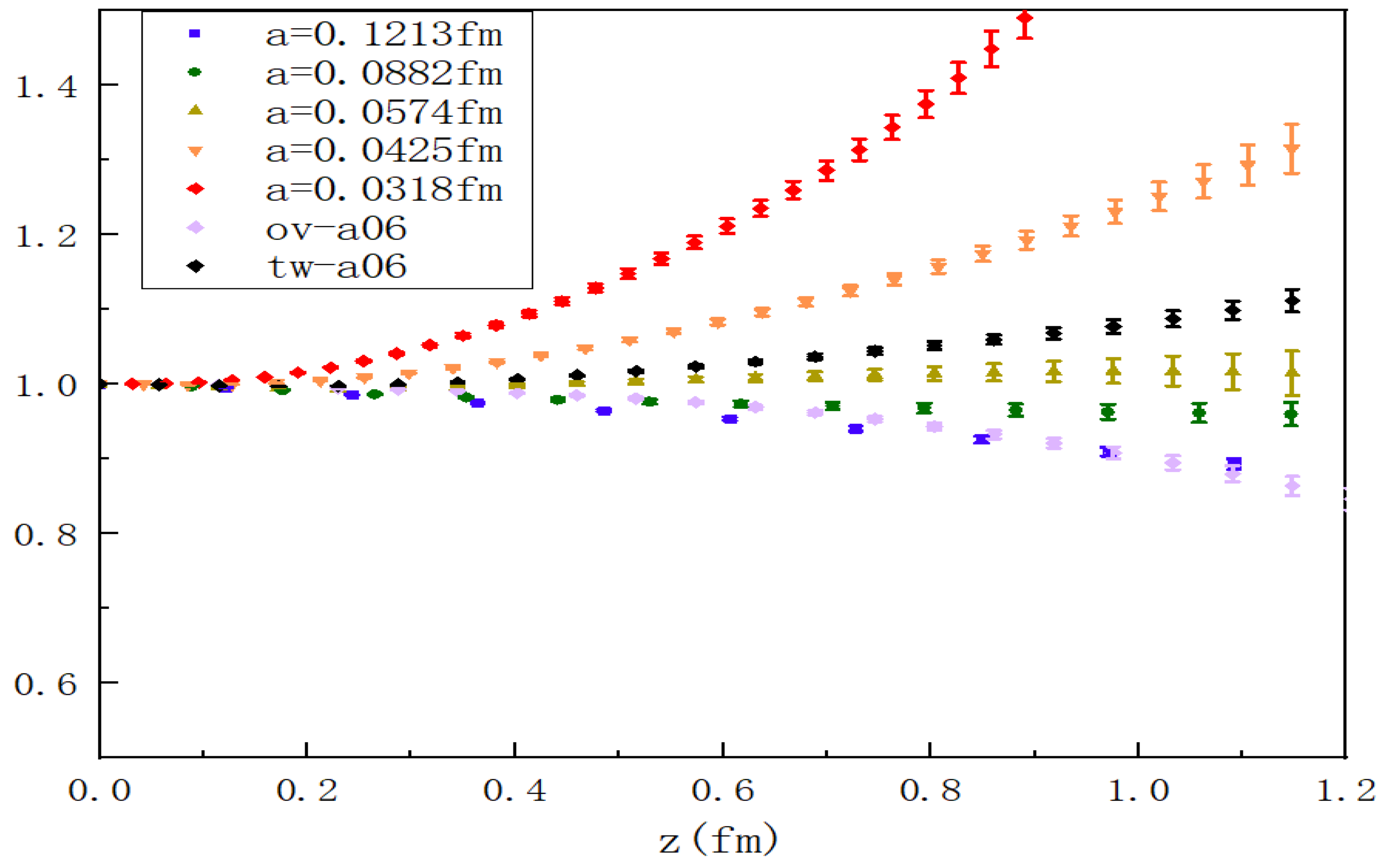
$$\frac{\mathrm{Tr}[\gamma_t \langle q | \mathcal{O}_t | q \rangle]_{p=(3,3,0,0)2\pi/L}}{\mathrm{Tr}[\gamma_t \langle q | \mathcal{O}_t | q \rangle]_{p=(5,5,0,0)2\pi/L}}$$

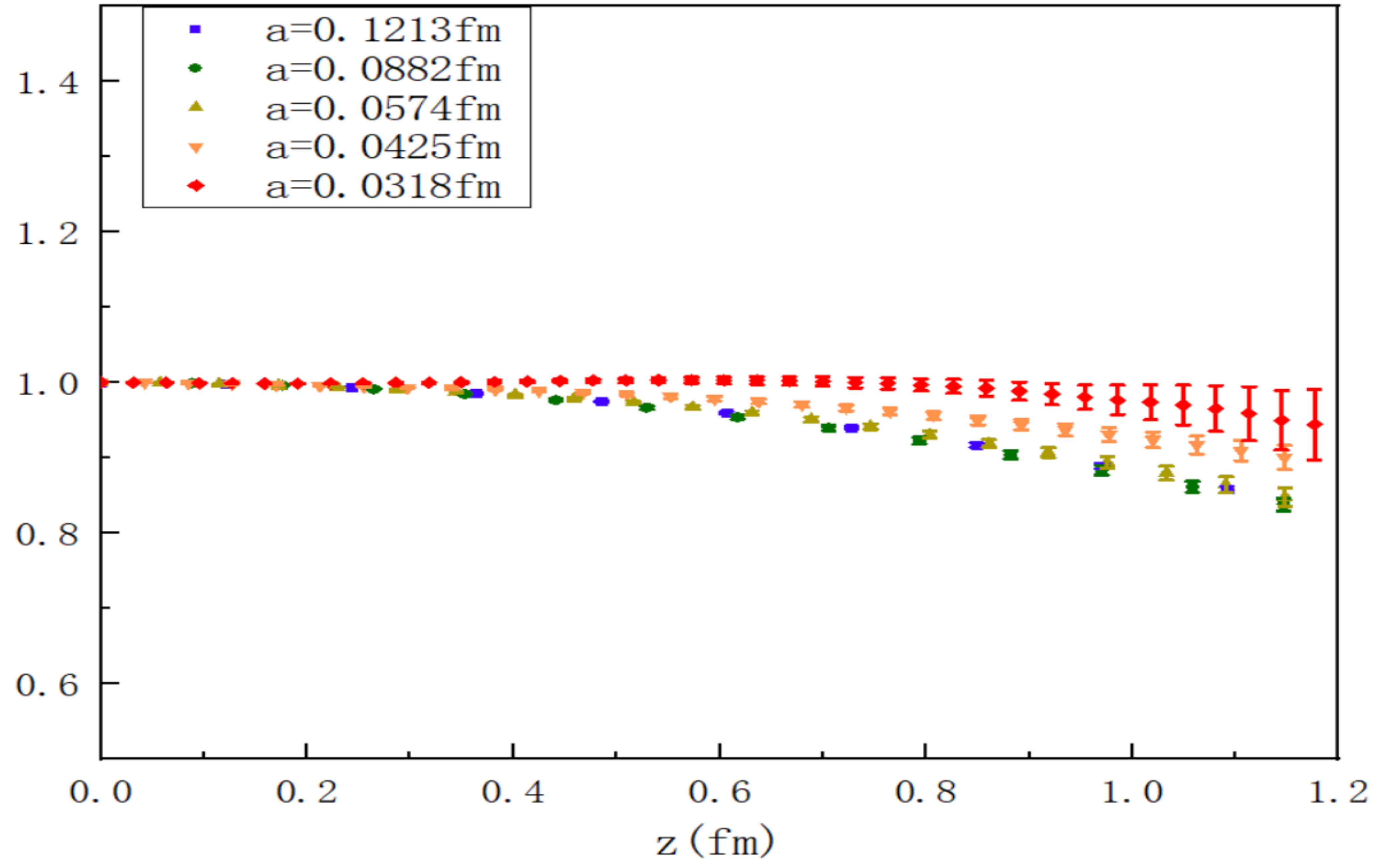


Quark matrix elements

Renormalized ME

- Renormalized ME: $\frac{\langle \pi | \mathcal{O}_t | \pi \rangle}{\mathrm{Tr}[\gamma_t \langle q | \mathcal{O}_t | q \rangle]_{p=(3,3,0,0)2\pi/L}}$.
- **All the cases will suffer from the residual linear divergence if the lattice spacing is small enough.**
- Upper panel: Clover on HISQ (Compare with the overlap on HISQ and Twisted-mass on HISQ cases at a~0.06 fm), about 24% residual linear divergence left.
- Lower panel: Overlap on HISQ, about 5% residual linear divergence.

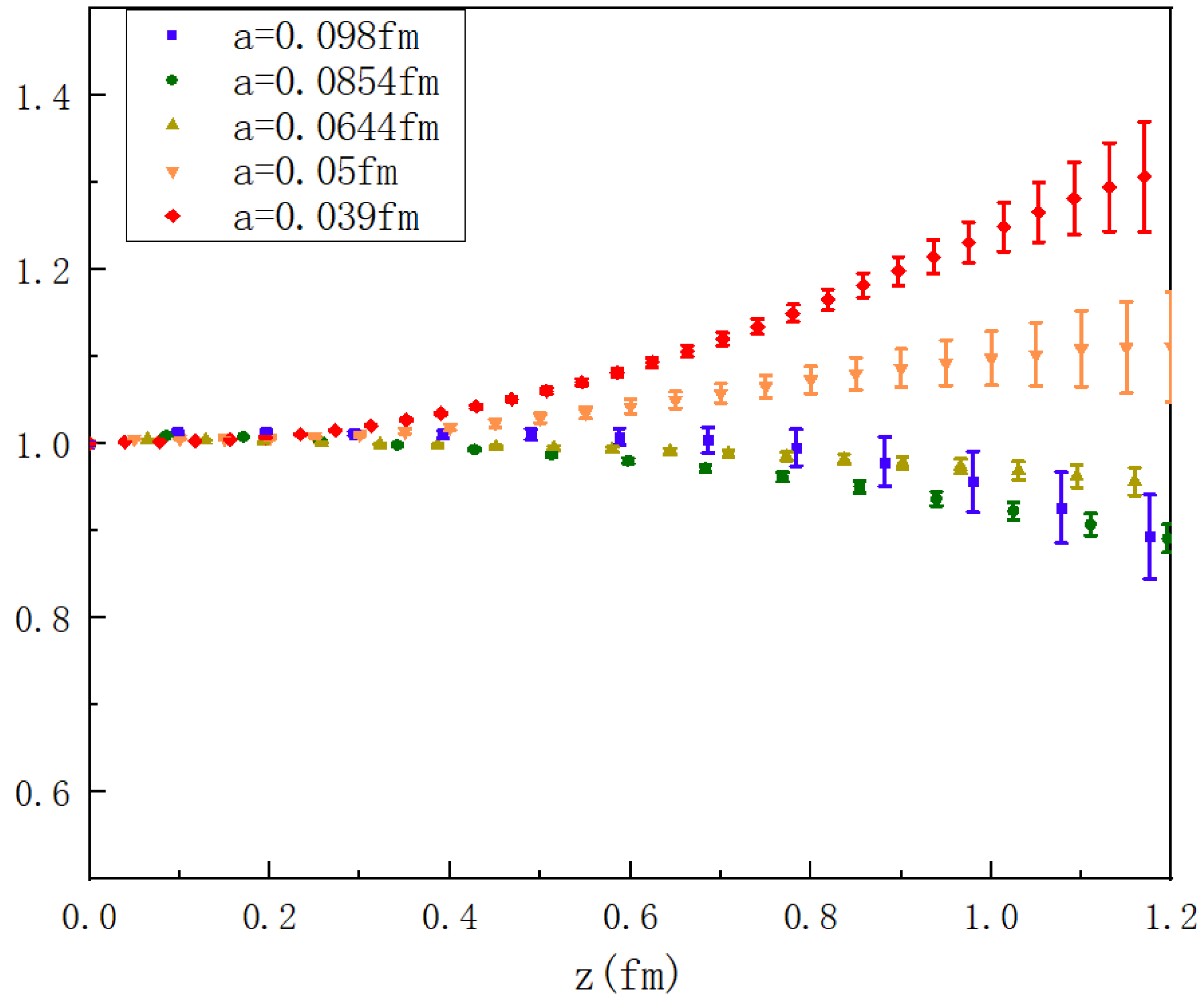




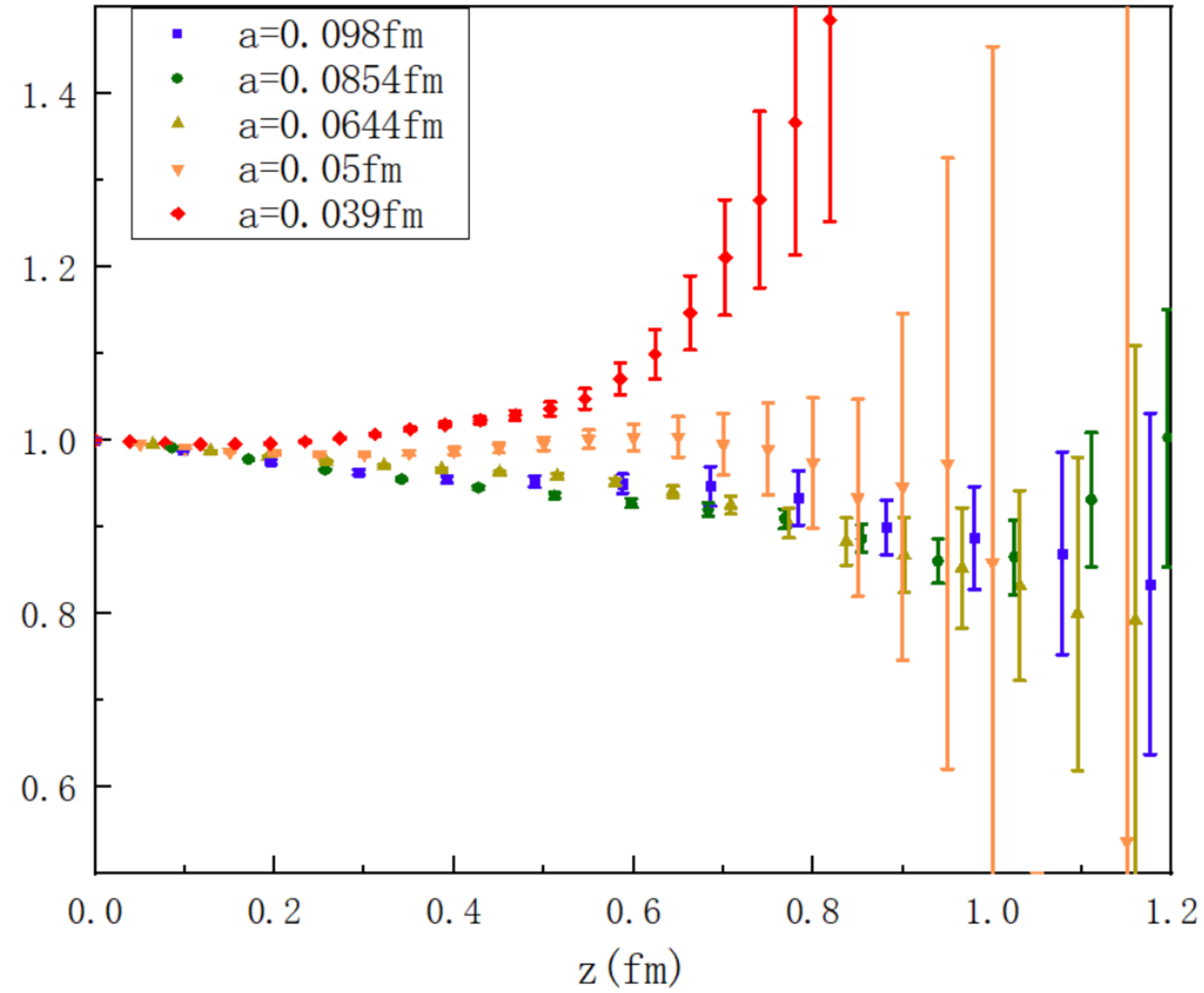
Quark matrix elements

Renormalized ME

- Renormalized ME: $\dfrac{\langle \pi | \mathcal{O}_t | \pi \rangle}{\mathrm{Tr}[\gamma_t \langle q | \mathcal{O}_t | q \rangle]_{p\sim 1.8\mathrm{GeV}}}$.
- **Unitary clover case using the CLS ensembles**, without any HYP smearing on the fermion action.
- In the range a~0.06-0.10 fm, the joint effect of the discretization error and residual linear divergence makes the results to be somehow similar to each other.



- With HYP smearing on the Wilson link
- About 18% residual linear divergence left.



- W.o. HYP smearing on the Wilson link
- The original linear divergence w.o. HYP smearing is much larger.
- And then the residual linear divergence is relatively smaller (about 5%).

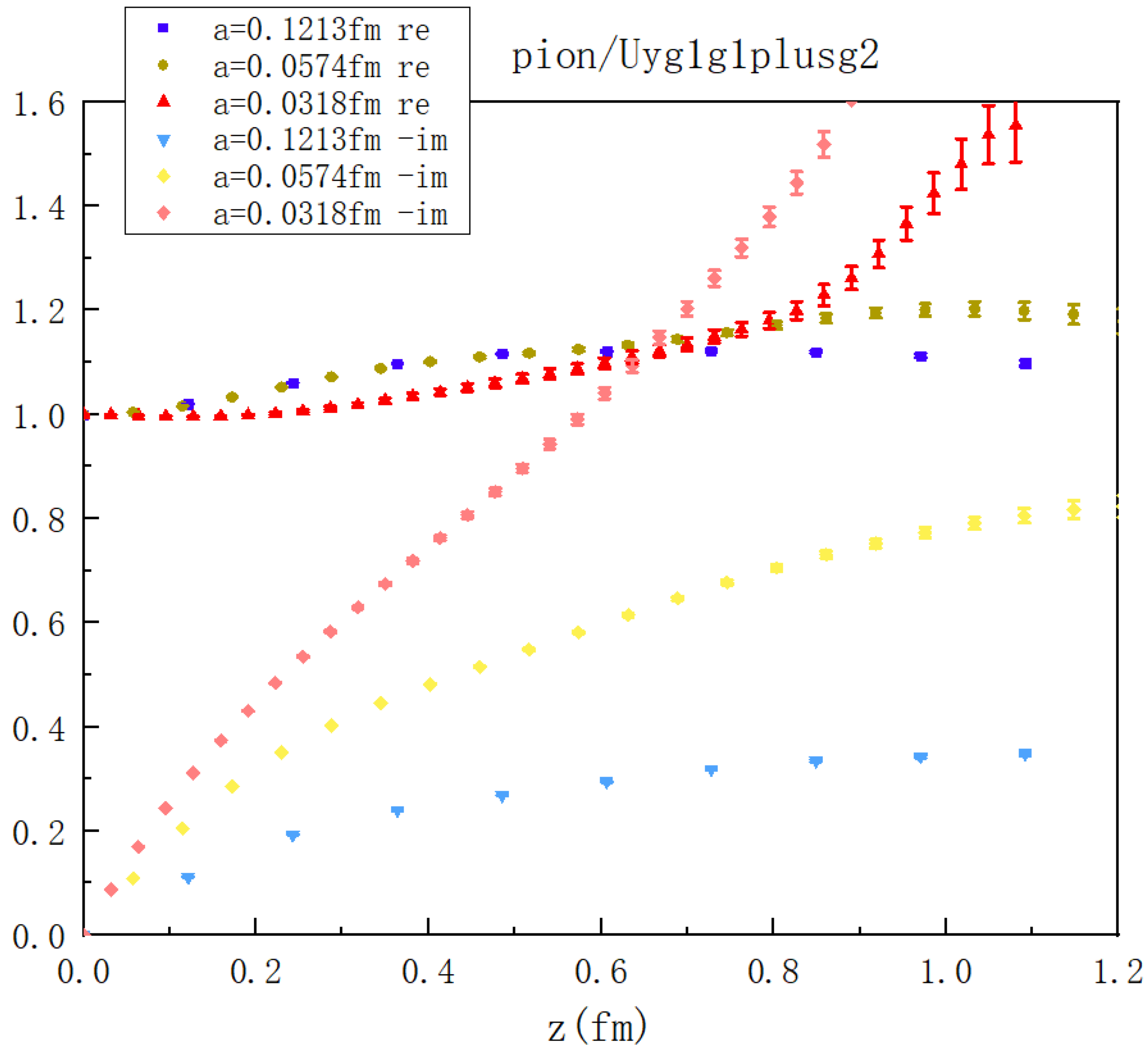
Quark matrix elements

Non-zero momentum case

- It is also interesting to consider another widely used renormalization $\dfrac{\langle \pi | \mathcal{O}_t | \pi \rangle}{\mathrm{Tr}[\not{p}\langle q | \mathcal{O}_t | q \rangle / p_t]_{p=(0,0,1.3,1.3)\mathrm{GeV}}}$.

- With the momentum along the Wilson link direction, the additional imaginary part is introduced.

- Most of the linear divergence can be moved to the imaginary part, in the z<0.7 fm region.

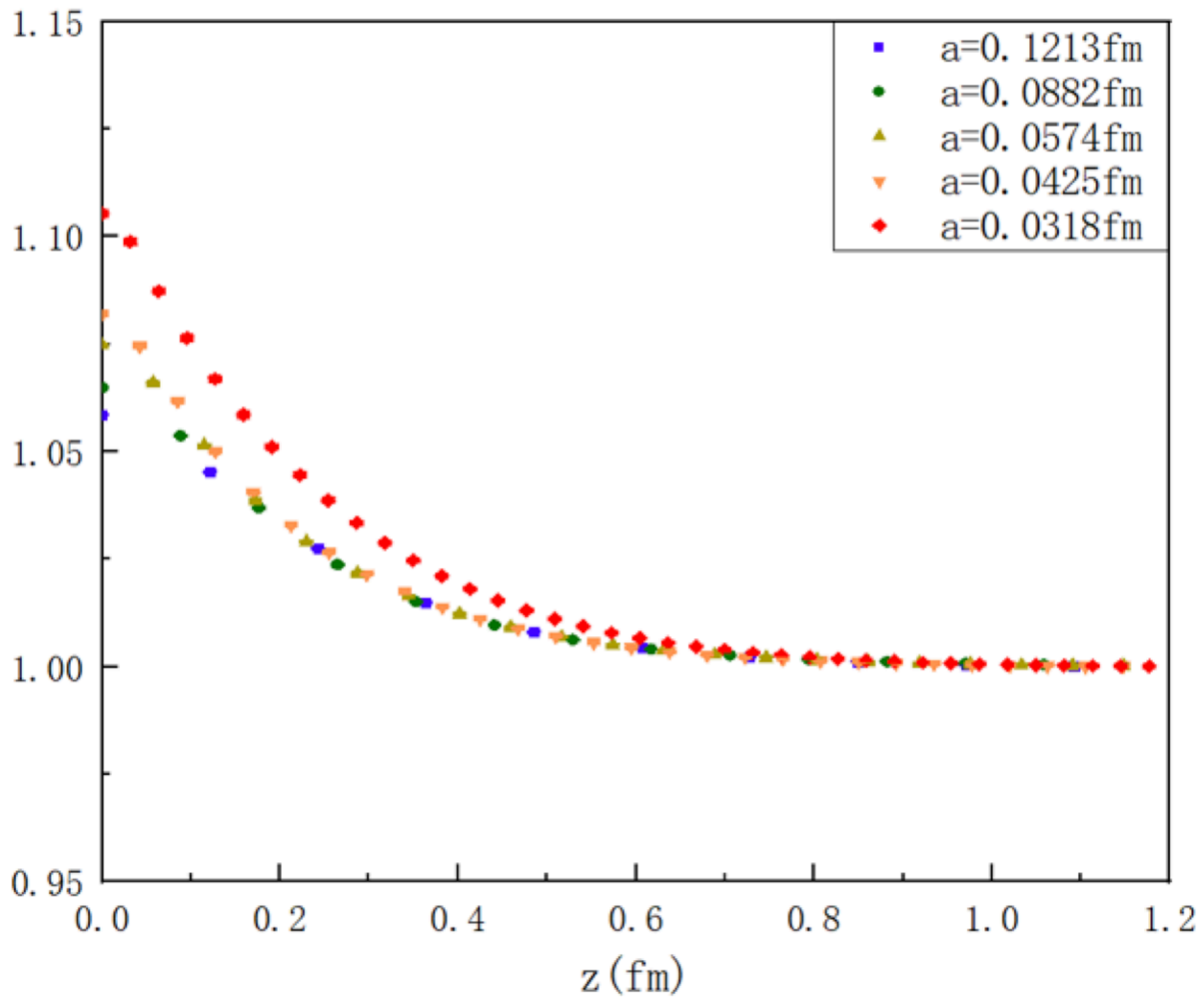


Quark matrix elements

Without link

$$\mathcal{M}_t^{(0)}(z) = \mathrm{Tr}[\gamma_t \langle S(p)\rangle^{-1} \langle \sum_x S^\dagger(p,x)\gamma_t S(p,x+\vec{n}_z z)\rangle \langle S(p)\rangle^{-1}]$$

- We can also consider a special case where the gauge link is removed from the quark matrix element.
- The matrix element approaches to 1 at the large z limit.
- The lattice spacing dependence at small z would be the logarithmic divergence.
- The residual linear divergence is not coming from here.



Outline

- Hadron matrix elements
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- **Discussion**

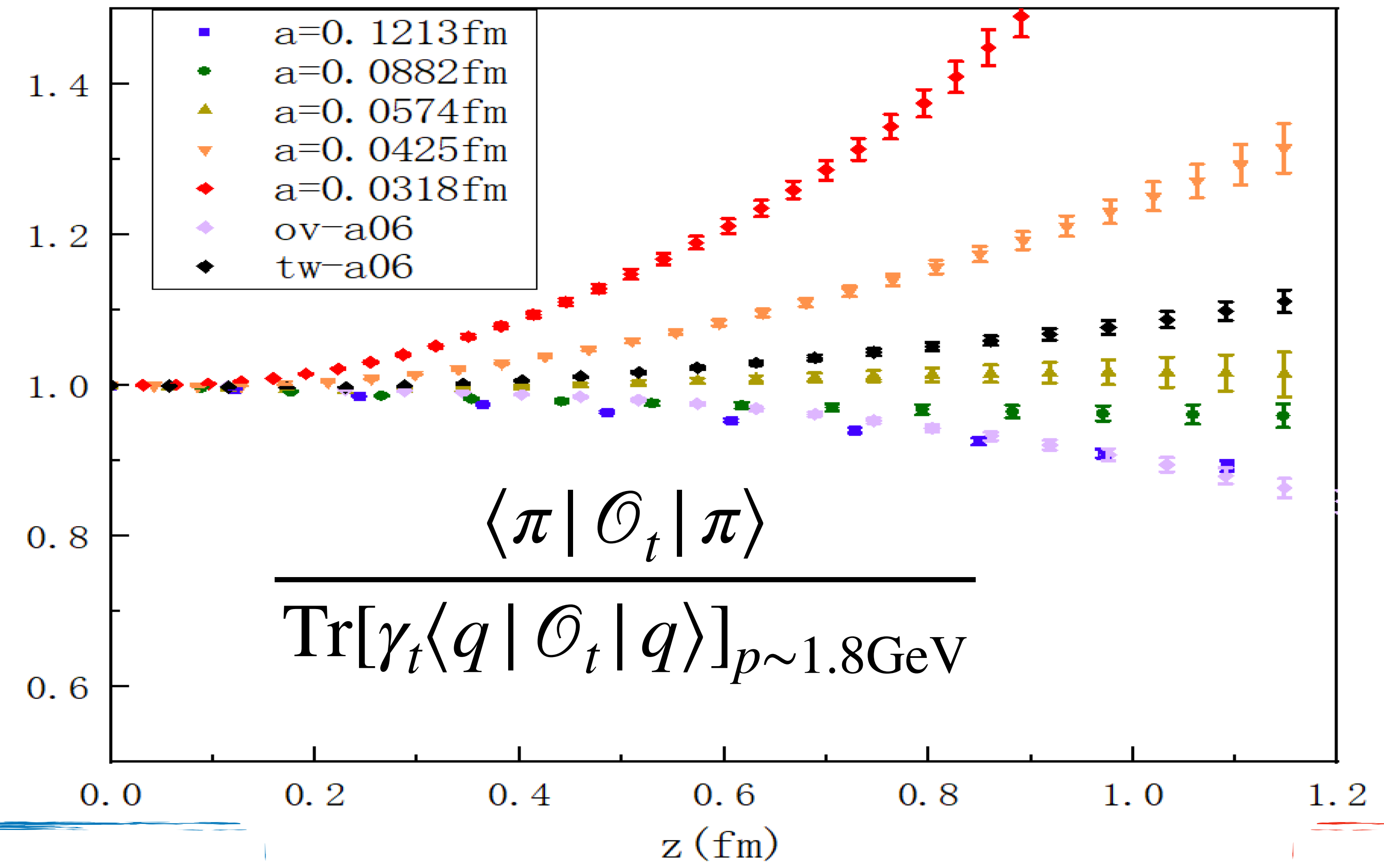
Discussion

Brief summary

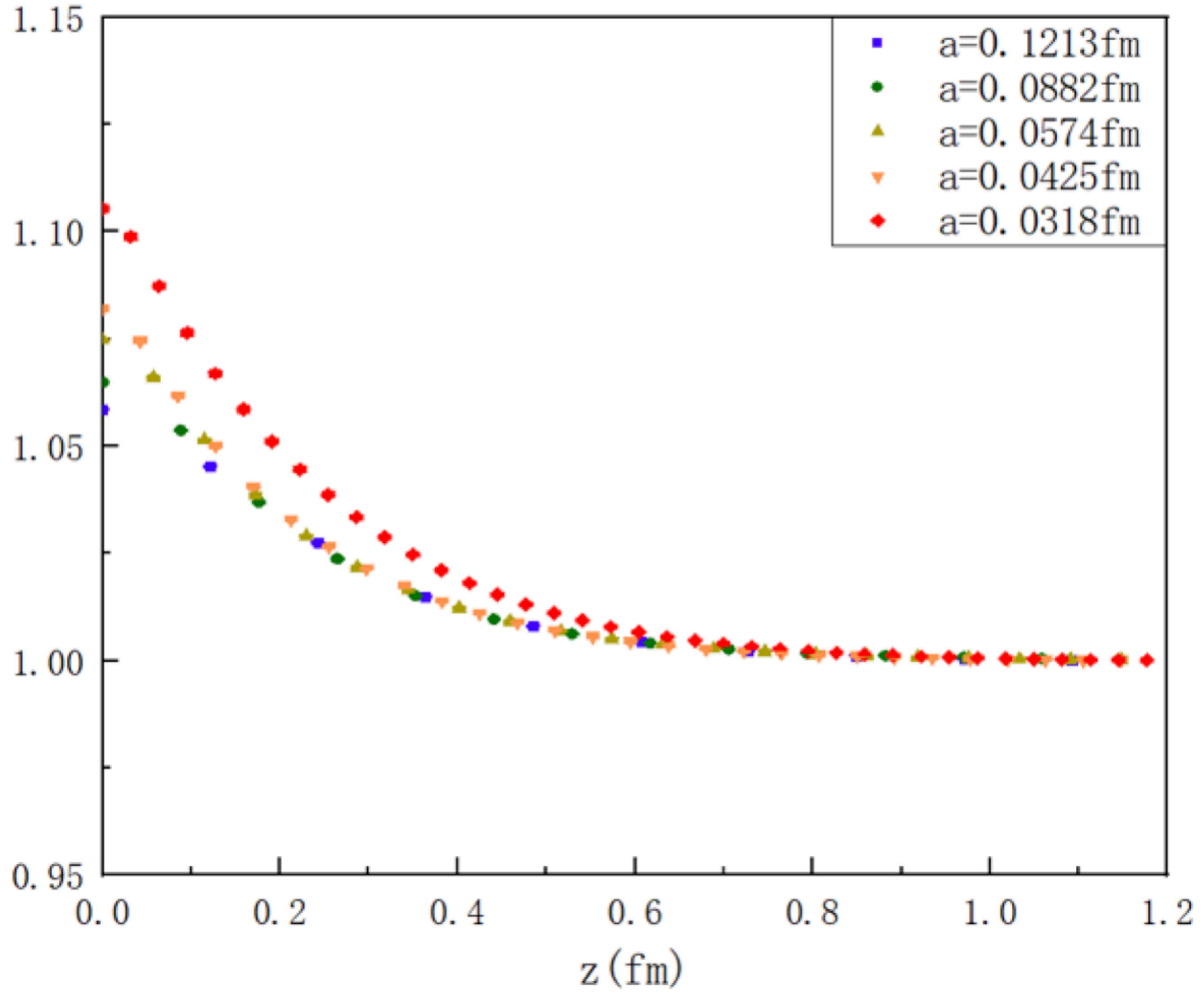
- Linear divergence in $\langle H | \mathcal{O}_t | H \rangle$ is independent to the fermion action or hadron, in the cases we checked.
- The linear divergence on the Wilson link would be independent with the gauge we used (Coulomb, Landau), in the continuum limit.
- The UV divergence of $\langle q | \mathcal{O}_t | q \rangle$ is independent from the IR momenta, but depends on the fermion action.
- The linear divergence disappear in $\langle q | \mathcal{O}_t | q \rangle$ if the Wilson link is removed.

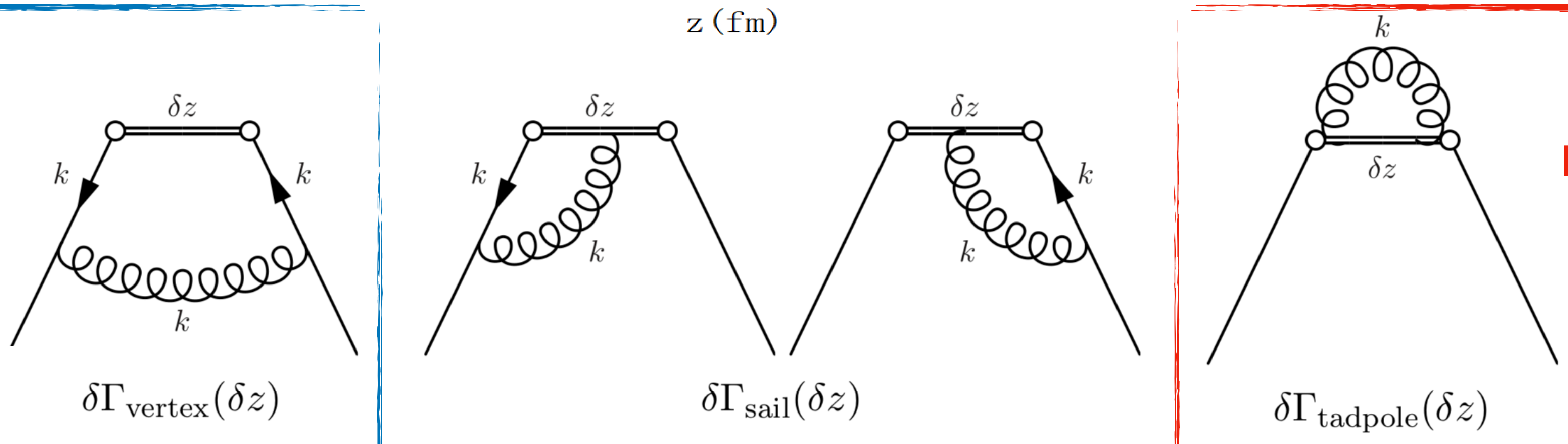
Discussion

Diagram analysis



No action-dependent linear divergence





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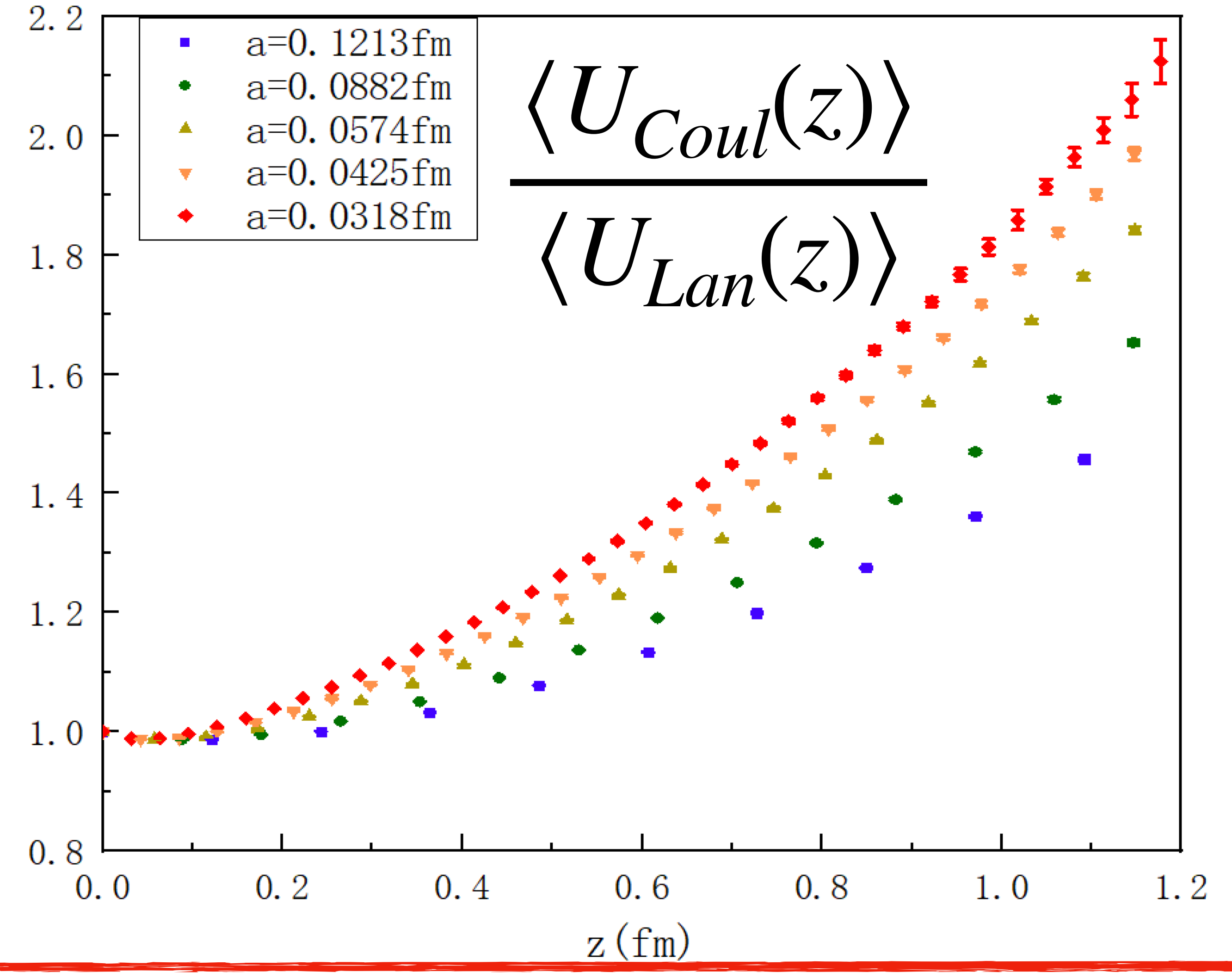
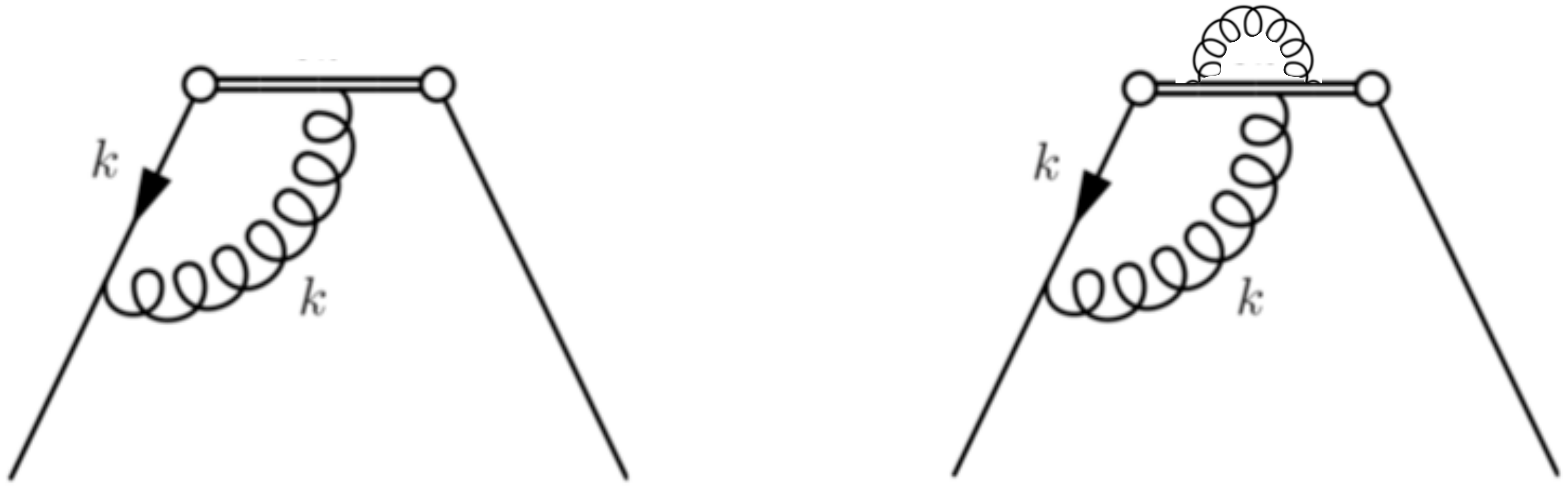


Discussion

Possibility reason

- Linear divergence in $\langle H|\mathcal{O}_t|H\rangle$ is independent to the fermion action or hadron, in the cases we checked.
- The linear divergence on the Wilson link would be independent with the gauge we used (Coulomb, Landau), in the continuum limit.
- The UV divergence of $\langle q|\mathcal{O}_t|q\rangle$ is independent from the IR momenta, but depends on the fermion action.
- The linear divergence disappear in $\langle q|\mathcal{O}_t|q\rangle$ if the Wilson link is removed.



- No linear divergence at 1-loop level;
- The external leg can have the action-dependent $\mathcal{O}(\alpha_s)$ correction.
- Thus some action-dependent $\mathcal{O}(\alpha_s^2/a)$ correction would exist.