Compton amplitude and the nucleon structure functions on the lattice via the Feynman-Hellmann theorem

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Lattice Hadron Structure Journal Club

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The numerical configuration generation (using the BQCD lattice QCD program)) and data analysis (using the Chroma software library) was carried out on the IBM BlueGene/Q and HP Tesseract using DIRAC 2 resources (EPCC, Edinburgh, UK), the IBM Blue-Gene/Q (NIC, Jülich, Germany) and the Cray XC40 at HLRN (The North-German Supercomputer Alliance), the NCI National Facility in Canberra, Australia (supported by the Australian Commonwealth Government) and Phoenix (University of Adelaide). RH is supported by STFC through grant ST/P000630/1. HP is supported by DFG Grant No. PE 2792/2-1. PELR is supported in part by the STFC under contract ST/G00062X/1. GS is supported by DFG Grant No. SCHI 179/8-1. KUC, RDY and JMZ are supported by the Australian Research Council grant DP190100297.

in Collaboration with

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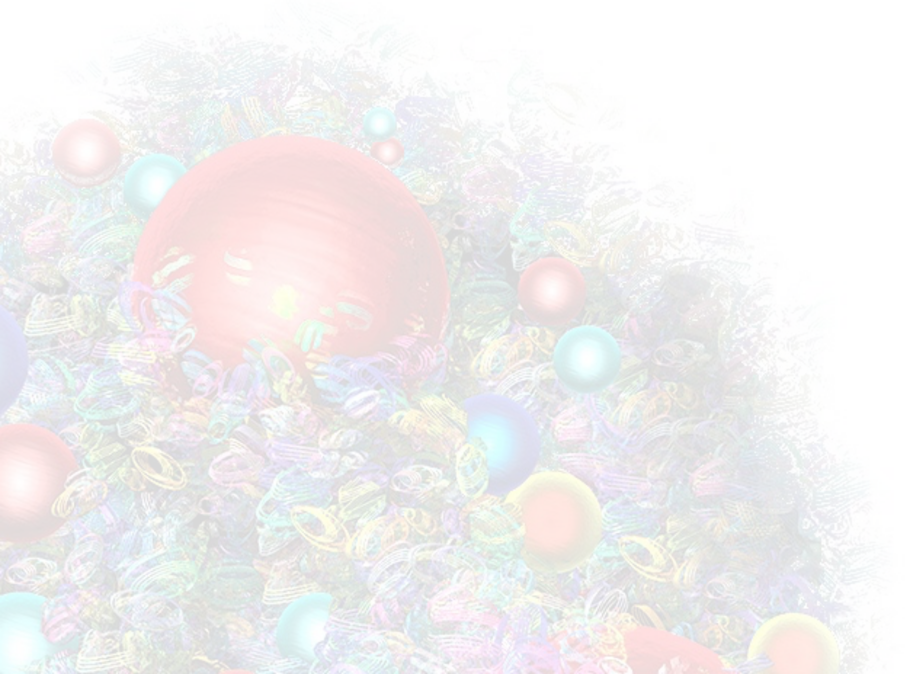
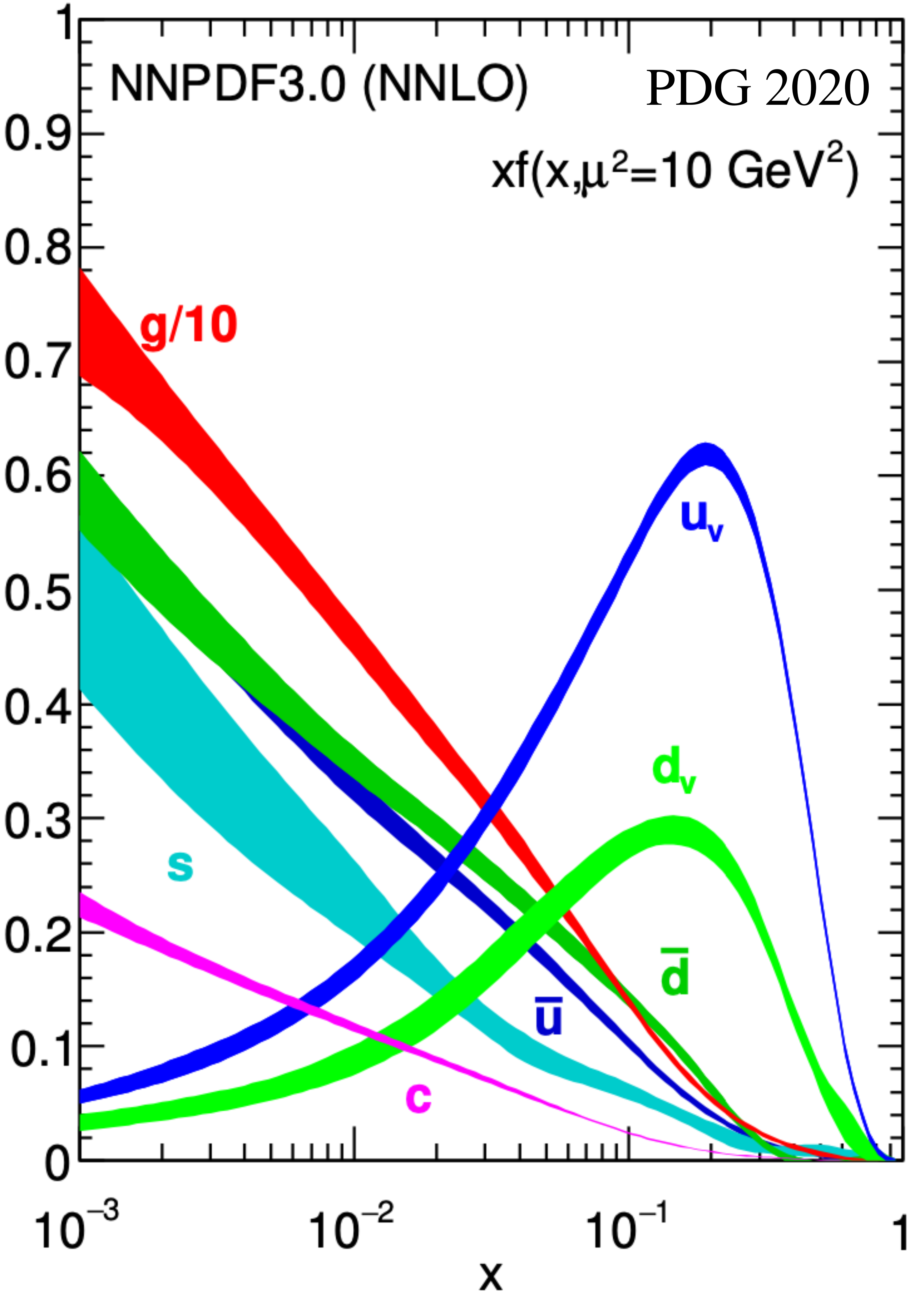


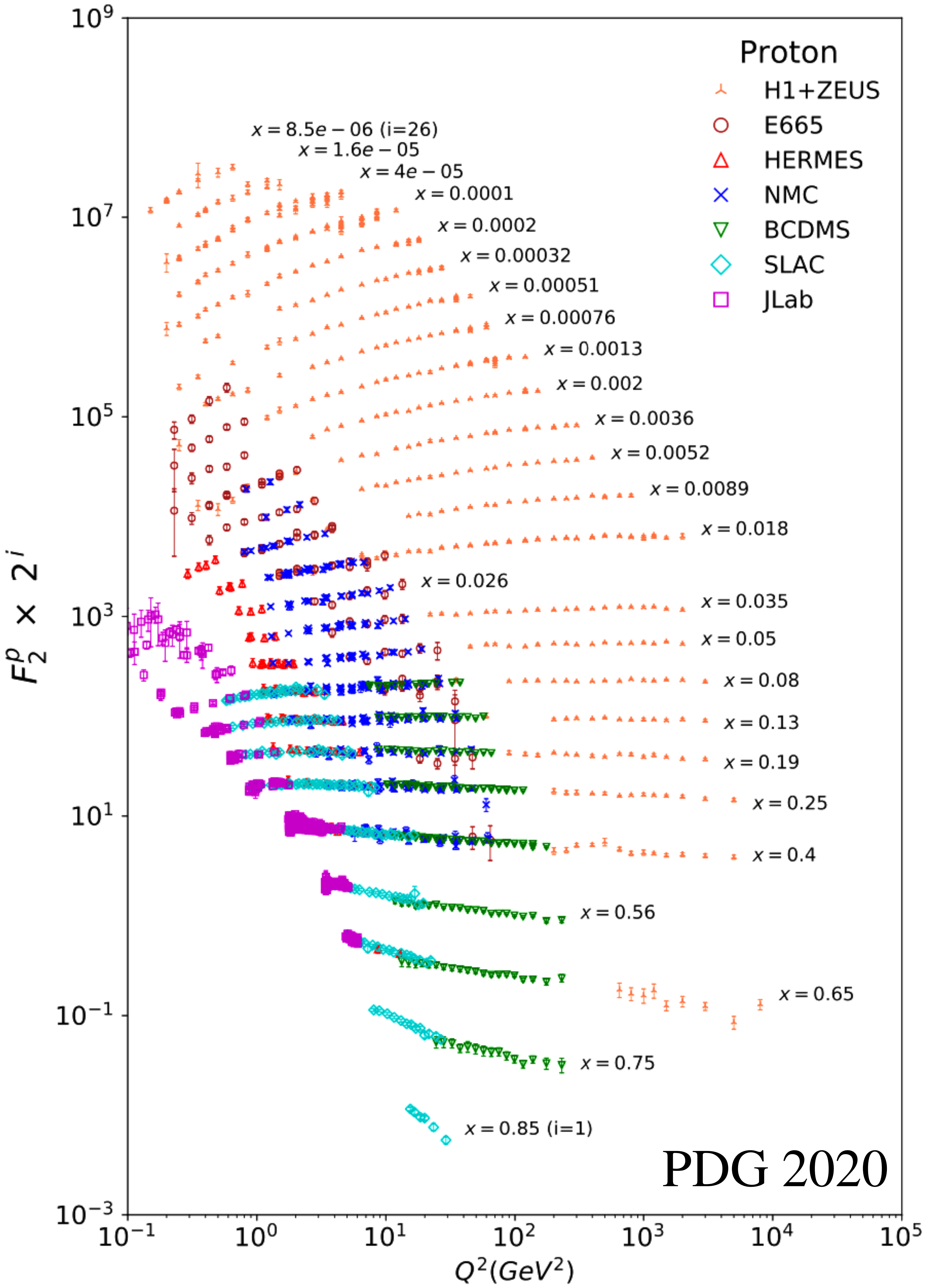
Motivation

- Nucleon structure (leading twist)
  - Structure functions from first principles
  - Understanding the behaviour in the high- and low-x regions



Motivation

- Scaling
  - Q^2 cuts of global QCD analyses
- Power corrections / Higher twist effects
  - Twist-4 contributions
  - Kinematic effects

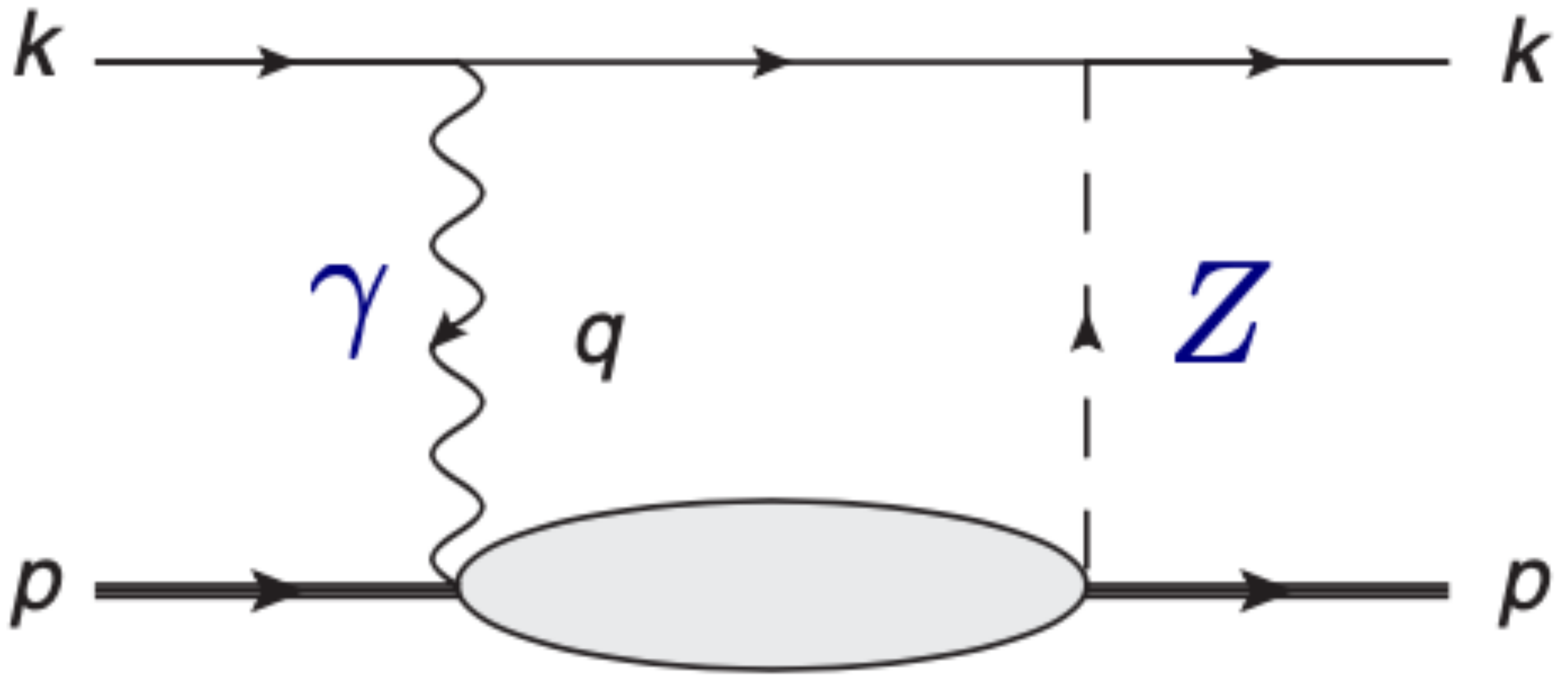


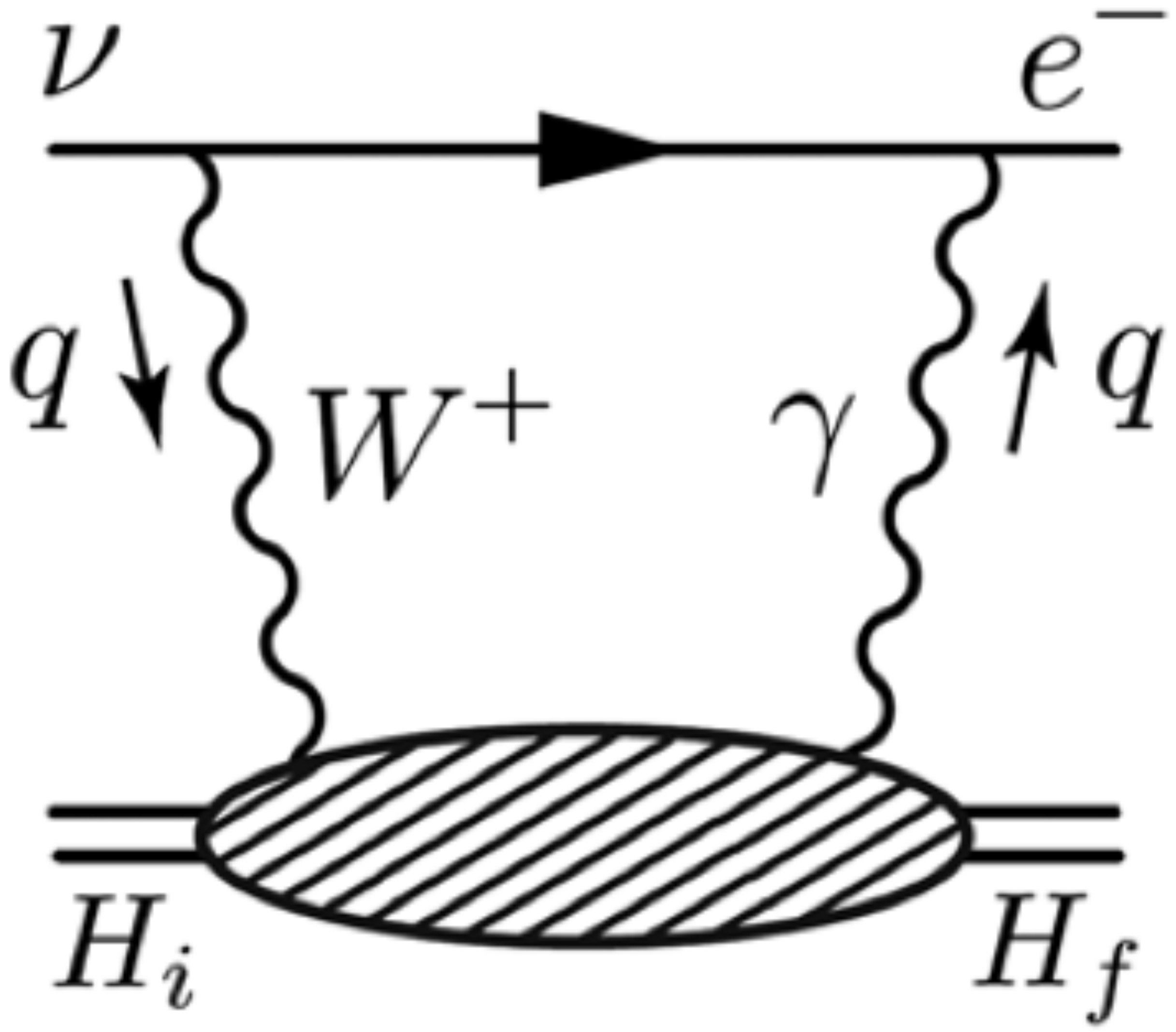
PDG 2020

Motivation

- New physics searches

  - Weak charge of the proton
  - $\gamma - W/Z$ interference





Motivation

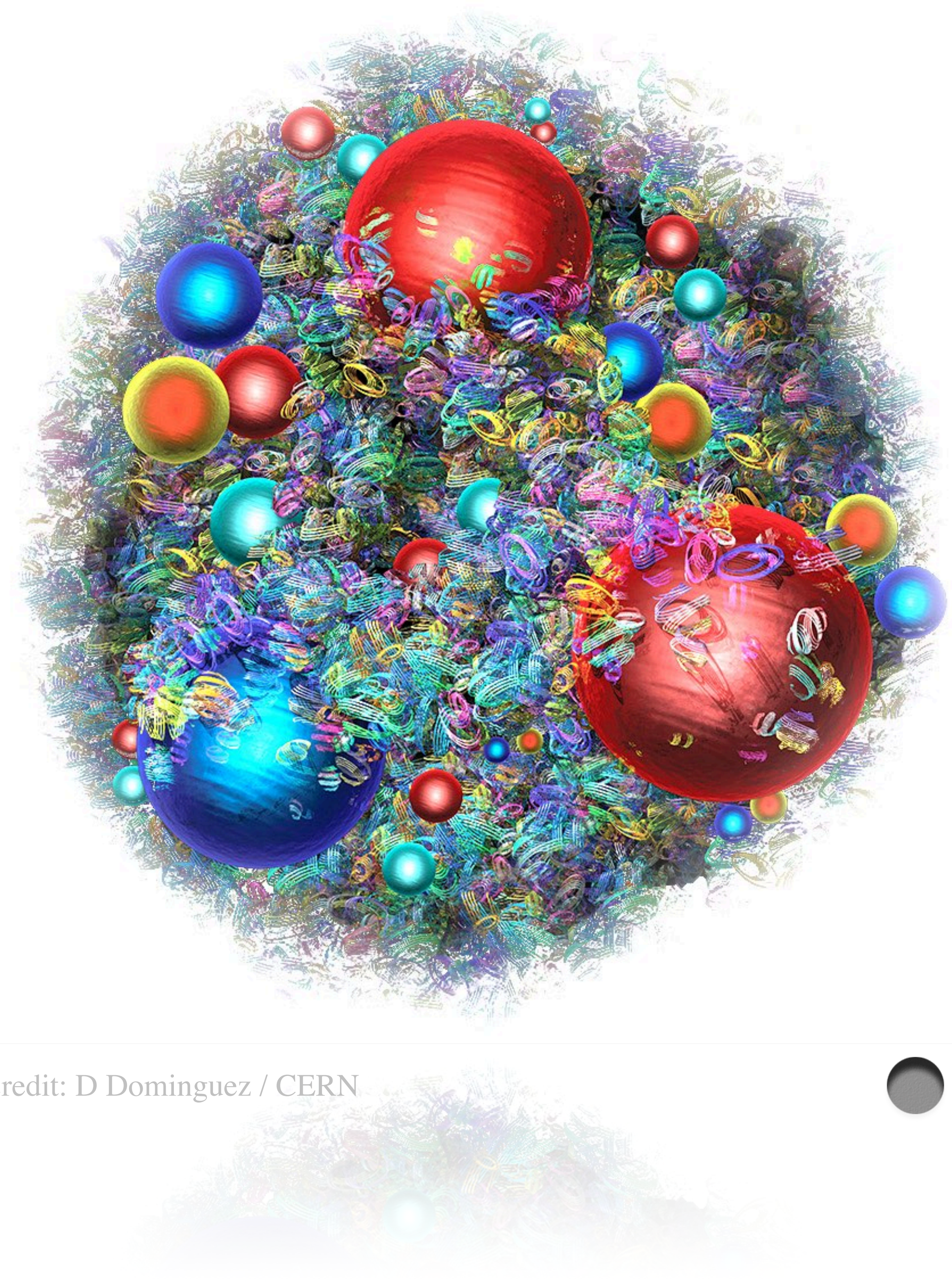
- Technical issues
  - Operator mixing/renormalisation issues in OPE approach in LQCD

$$\mu(Q^2) = \underset{1/a^2\text{ divergence}}{c_2(a^2Q^2)}\,\overset{\text{twist-2}}{\boxed{v_2(a)}} + \frac{c_4(a^2Q^2)}{Q^2}\,\overset{\text{twist-4}}{\boxed{v_4(a)}} + \cdots$$

mixing

  - 4-point functions are costly; harder to tackle
  - Feynman-Hellmann approach needs 2-point functions only

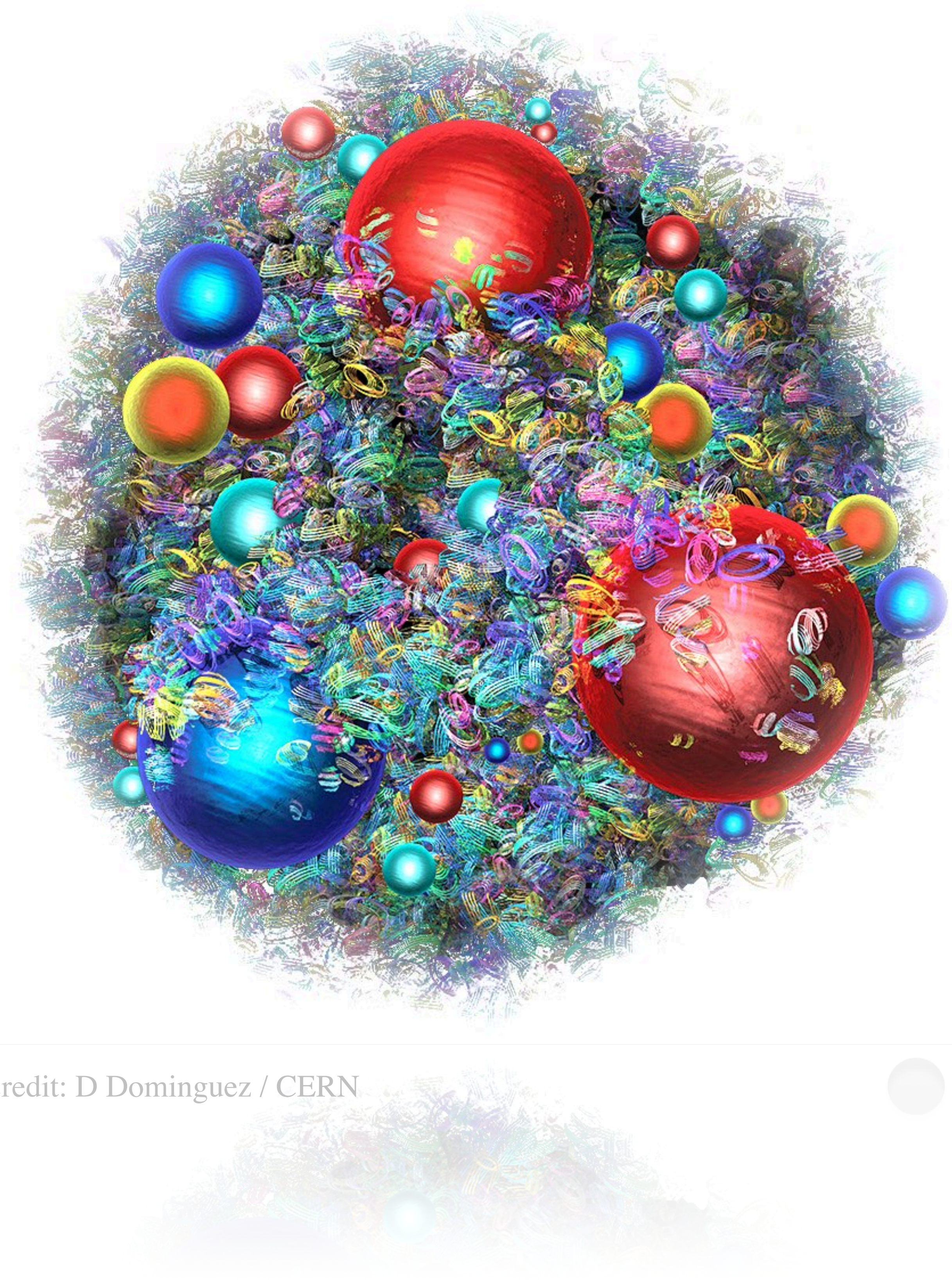
Outline



Credit: D Dominguez / CERN

- Forward Compton Amplitude & the Nucleon Structure Functions
- Feynman-Hellmann Theorem & the Compton Amplitude
- Moments of the Nucleon Structure Functions
- Scaling and Power Corrections/Higher-twist effects

Outline


Credit: D Dominguez / CERN

- Forward Compton Amplitude & the Nucleon Structure Functions
  - Feynman-Hellmann Theorem & the Compton Amplitude
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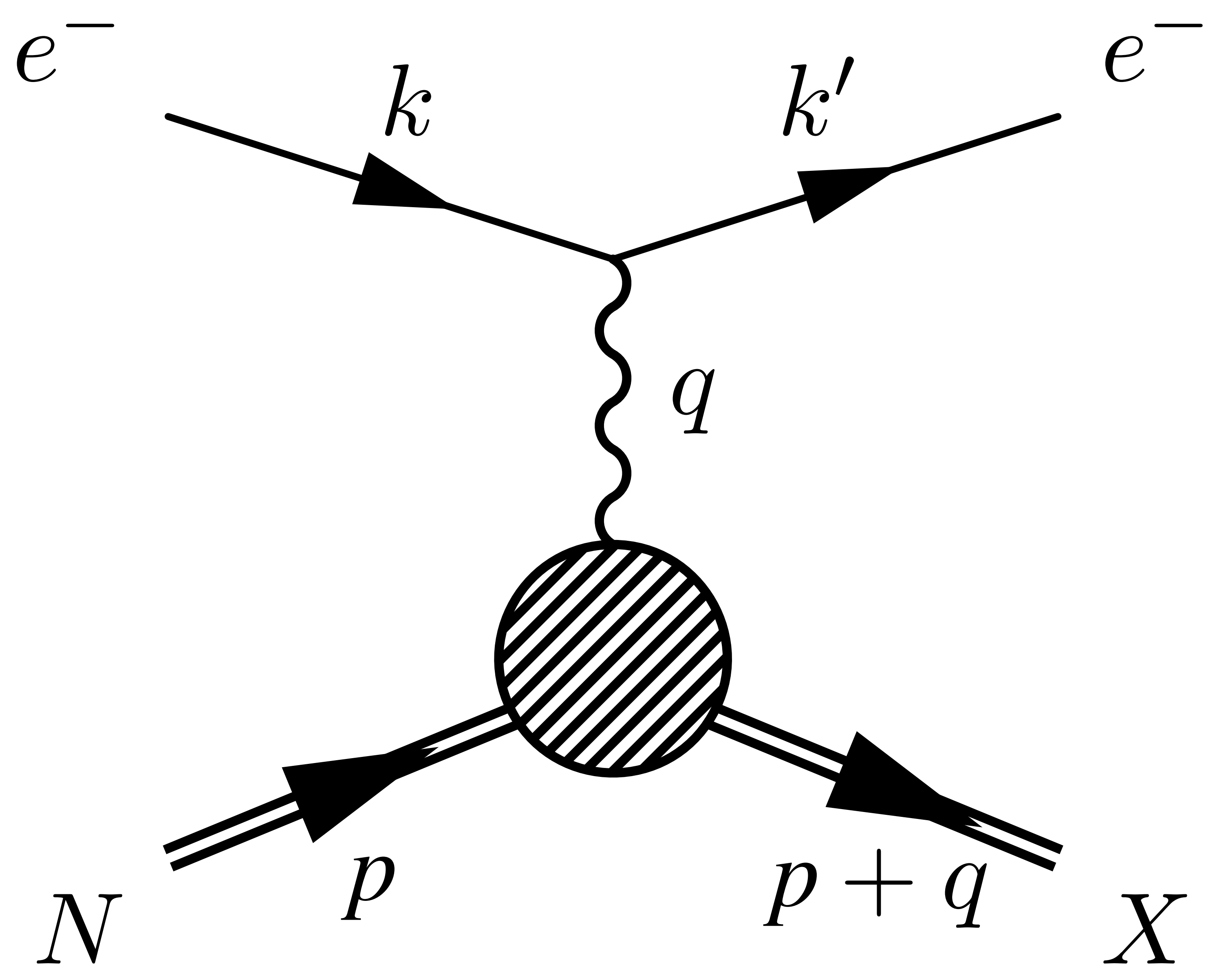
DIS and the Hadronic Tensor

Deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering (DIS)



- k, k': incoming, outgoing lepton momenta
- $\boldsymbol{p}$: 4-momentum of the incoming nucleon of mass $\boldsymbol{M}$
- $W^2 = (p+q)^2$: invariant mass of the recoiling system, $\boldsymbol{X}$
- $x = \dfrac{Q^2}{2p \cdot q}$: Bjorken scaling variable
- $\omega = x^{-1}$: inverse Bjorken variable
- $Q^2 = -q^2$: photon virtuality,
  momentum transferred to the nucleon

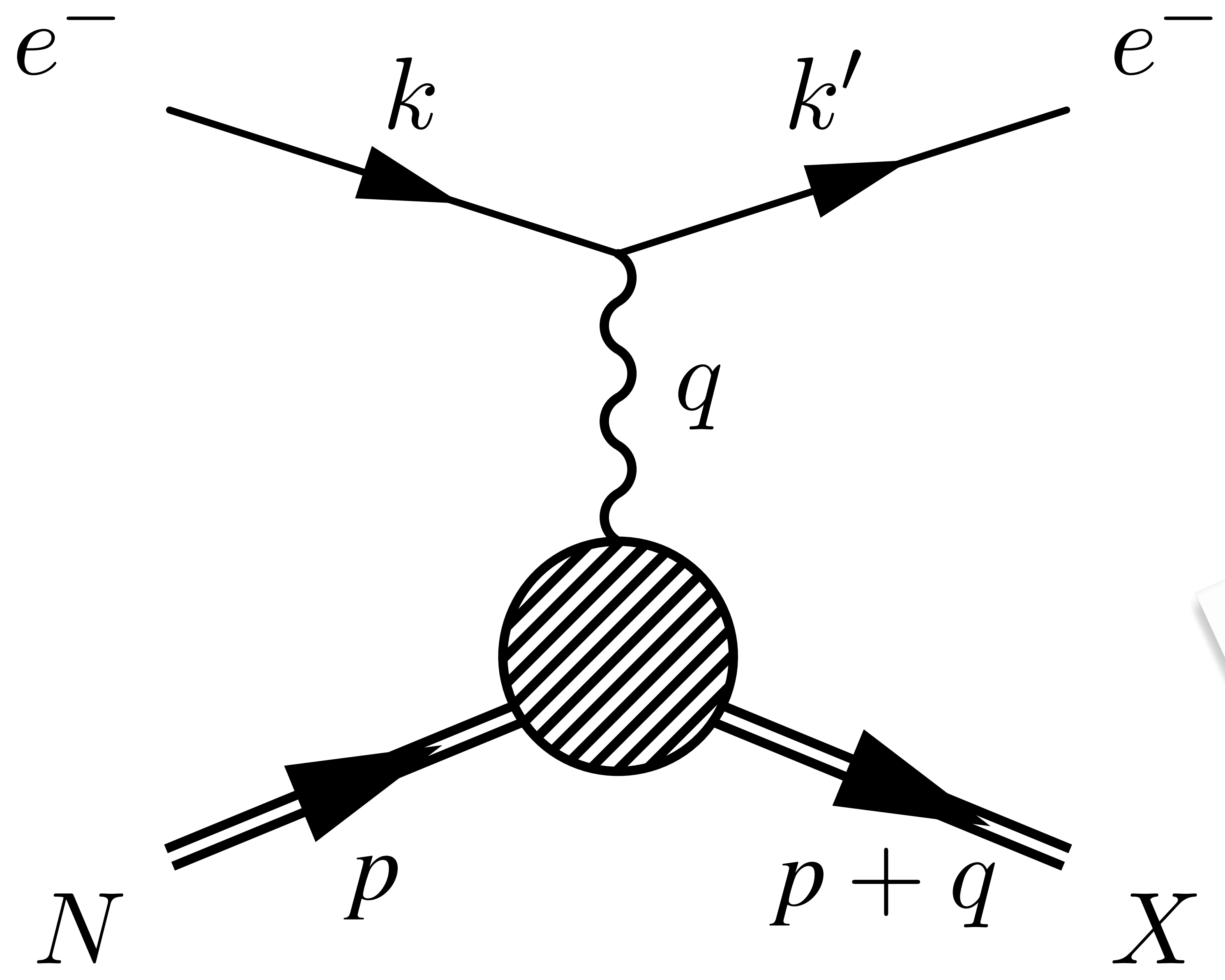
DIS and the Hadronic Tensor

Deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering (DIS)

$$d\sigma \sim L_j^{\mu\nu} W_{\mu\nu}^{j}$$

$j = \gamma, Z,$ and γZ (neutral) or $\boldsymbol{W}$ (charged)

leptonic tensor ***hadronic tensor***



$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4 z e^{iq\cdot z}\rho_{ss'}\langle p,s'|[J_\mu(z),J_\nu(0)]|p,s\rangle$$

$$\rho_{ss'} = \tfrac{1}{2}\delta_{ss'}$$

unpolarised

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)F_1(x,Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\frac{F_2(x,Q^2)}{p\cdot q}$$

Structure Functions

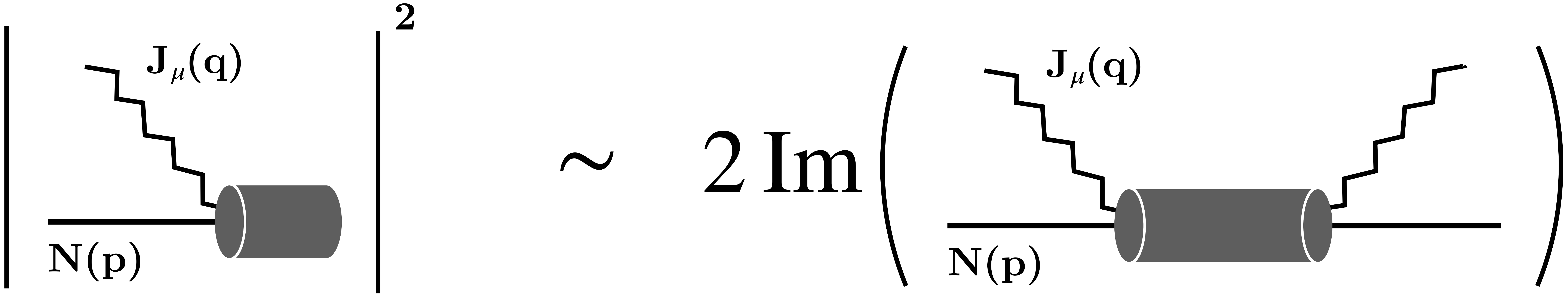
Forward Compton Amplitude

$$\omega = \frac{2p\cdot q}{Q^2}$$

$$T_{\mu\nu}(p,q) = i\int d^4z\, e^{iq\cdot z}\rho_{ss'}\langle p,s'|\mathscr{T}\{J_\mu(z)J_\nu(0)\}|p,s\rangle \quad \text{, spin avg. } \rho_{ss'} = \frac{1}{2}\delta_{ss'}$$

$$= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\mathscr{F}_1(\omega,Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\frac{\mathscr{F}_2(\omega,Q^2)}{p\cdot q}$$

Compton Structure Functions (SF)



DIS Cross Section ~ Hadronic Tensor

Forward Compton Amplitude ~ Compton Tensor

Nucleon Structure Functions

- **Consider:**

$\mu = \nu = 3$ and $p_z = q_z = 0$

$$T_{33}(p,q) = \mathscr{F}_1(\omega, Q^2)$$

- **Optical theorem relates the Compton SF to DIS SF**

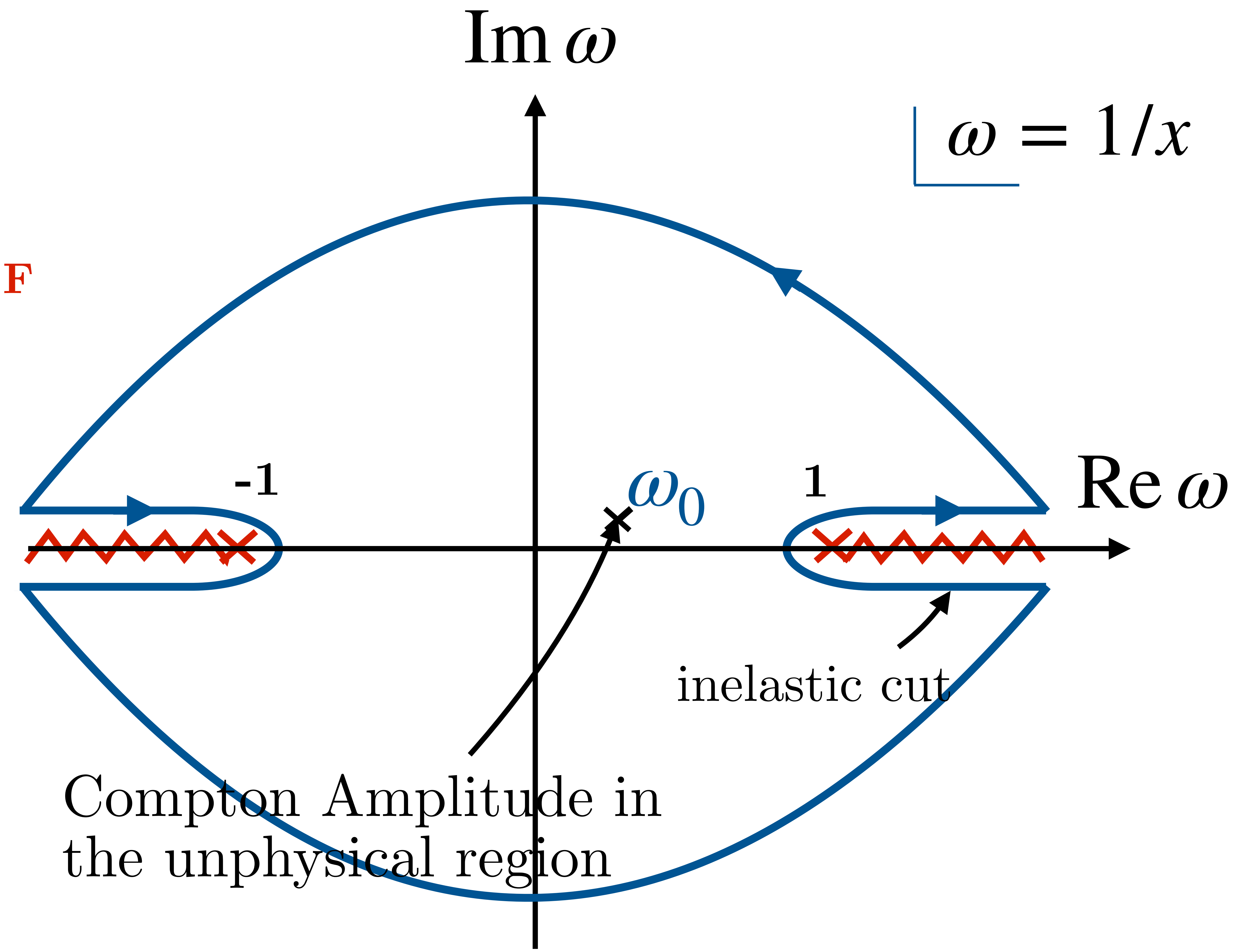
$$\mathrm{Im}\,\mathscr{F}_1(\omega, Q^2) = 2\pi F_1(x, Q^2)$$

so we can write down a dispersion relation:

$$\overline{\mathscr{F}}_1(\omega, Q^2) = \frac{2\omega^2}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\,\mathscr{F}_1(\omega', Q^2)}{\omega'\left(\omega'^2 - \omega^2 - i\epsilon\right)}$$

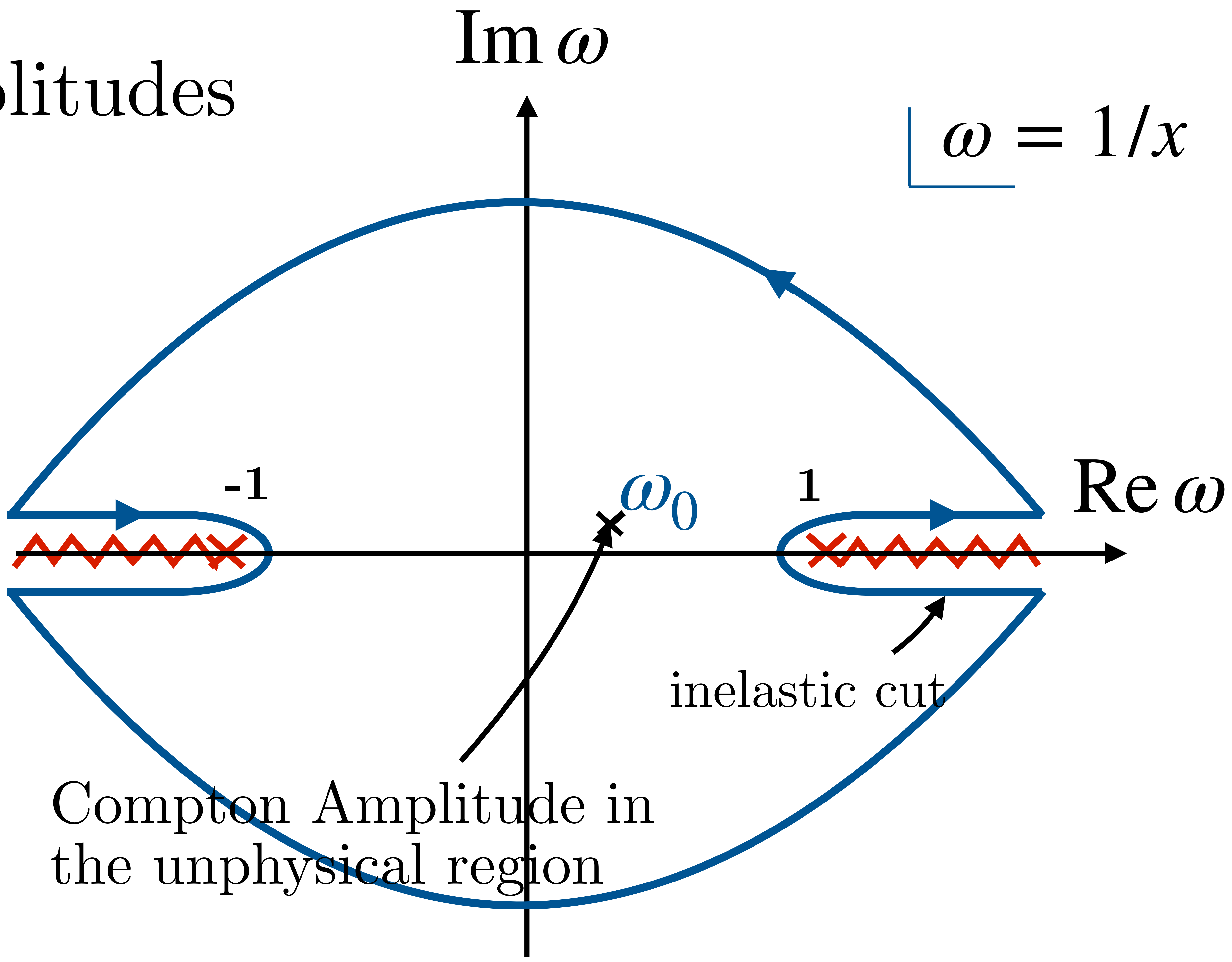
subtracted
dispersion relation

$$= 4\omega^2 \int_0^1 dx \frac{x\,F_1(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

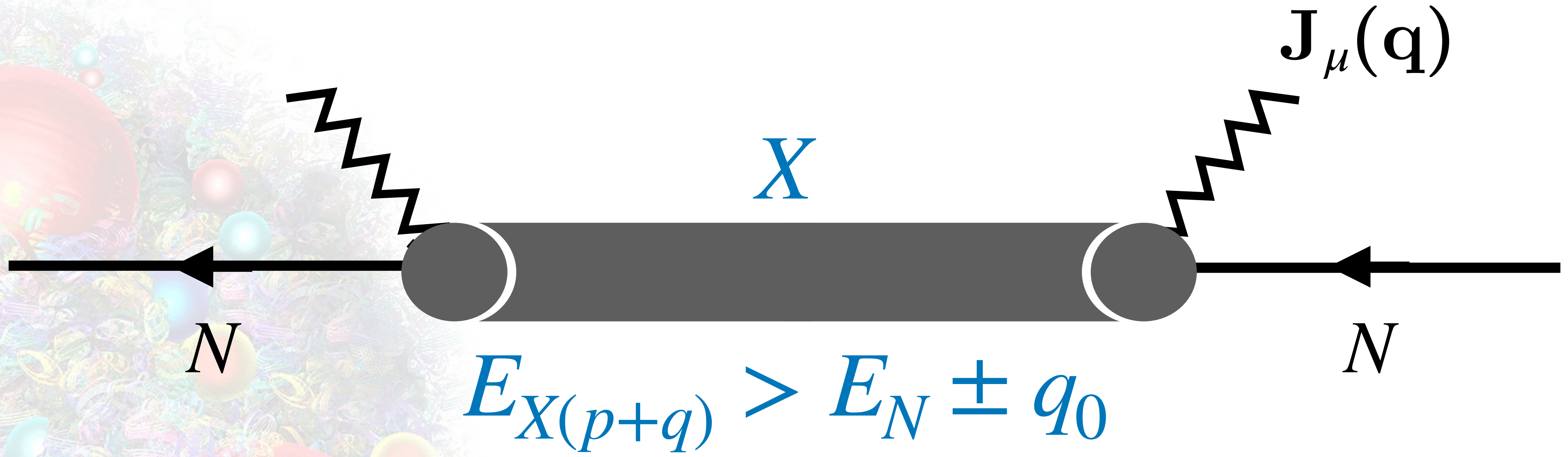


Nucleon Structure Functions

- As long as $|\omega_0| < 1$, Minkowski and Euclidean amplitudes are identical
- $|\omega_0| < 1$ means states propagating between currents <u>cannot go on-shell</u>



Compton Amplitude in the unphysical region



$E_{X(p+q)} > E_N \pm q_0$

Nucleon Structure Functions

$$\omega = \frac{2p \cdot q}{Q^2}$$

Compton amplitude with
$\mu = \nu = 3$ and $p_z = q_z = 0$

$$T_{33}(p,q) = \mathscr{F}_1(\omega, Q^2)$$ Compton SF

$$\overline{\mathscr{F}}_1(\omega, Q^2) = 4\omega^2 \int_0^1 dx \frac{x\, F_1(x, Q^2)}{1 - x^2\omega^2}$$

subtracted
dispersion relation

we are at the unphysical $|\omega| < 1$ region, no need for $i\varepsilon$

Taylor expand $[1-(xw)^2]^{-1}$

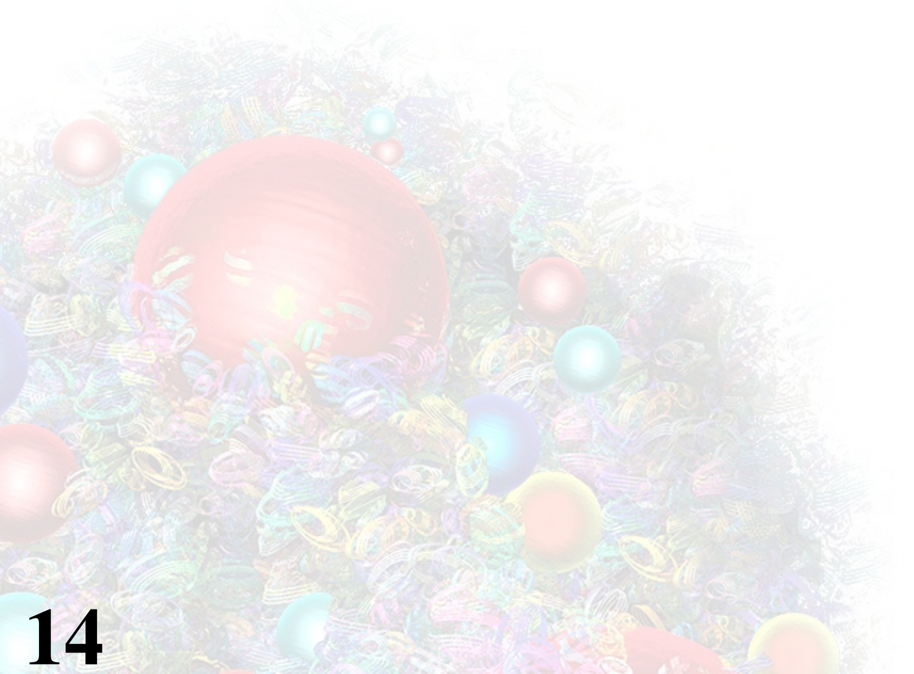
$$= \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$$

, **where** $M_{2n}^{(1)}(Q^2) = 2\int_0^1 dx\, x^{2n-1} F_1(x, Q^2)$

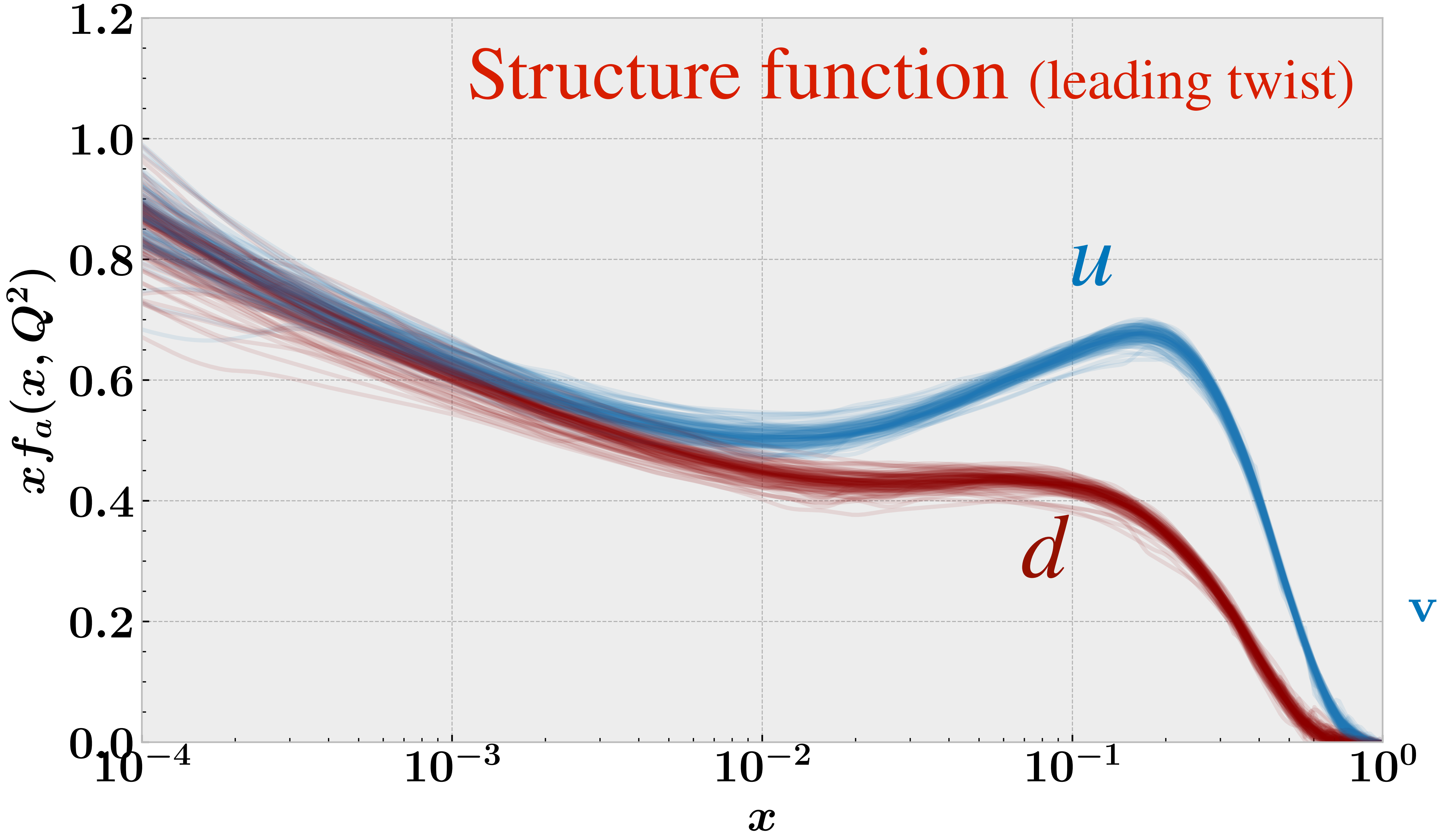
Mellin moments of the nucleon structure function $F_1(x, Q^2)$

Once we have the Compton amplitude data, we can extract the Mellin moments!

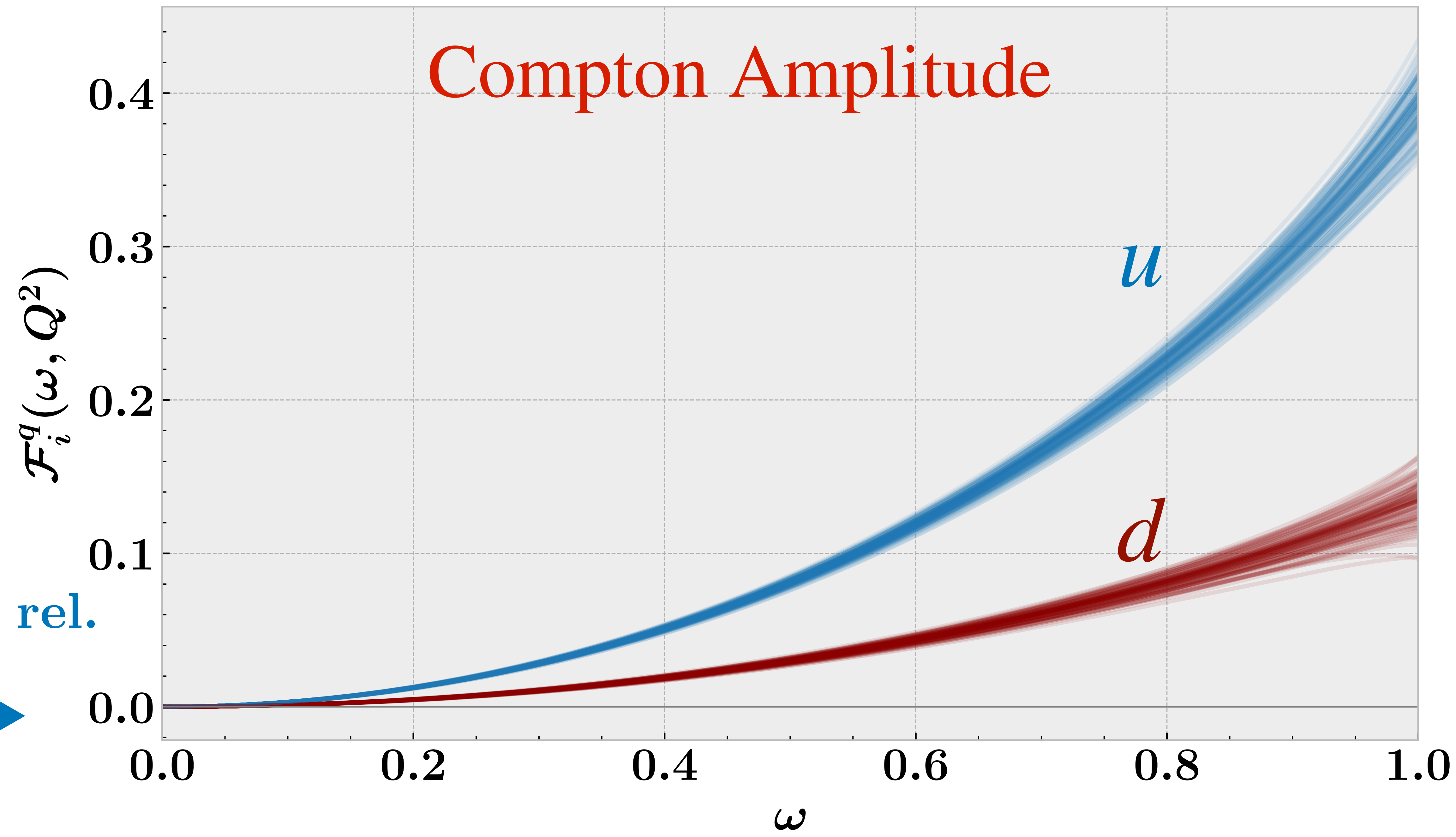
$$T_{33}(p,q) = \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$$



Shape of the Compton Amplitude



via disp. rel.



NNPDF3.1 NNLO

100 sets

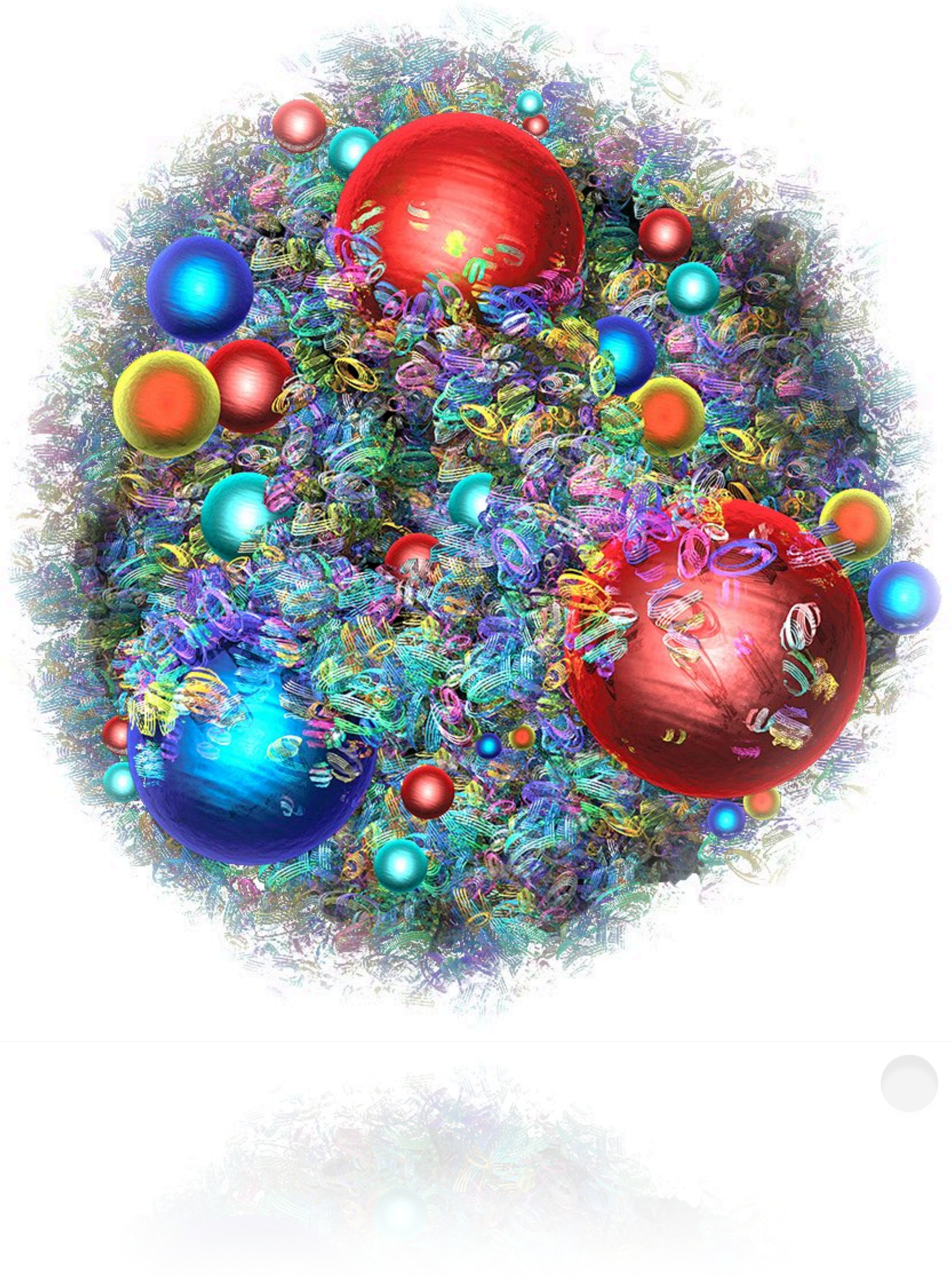
$Q^2 = 9\,GeV^2$

(DIS region)

$$T_{33}(p,q) = \sum_{n=1}^{\infty} 2\omega^{2n} \boxed{M_{2n}^{(1)}(Q^2)}$$

Moments
of the DIS
Structure Functions

Outline



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FH Theorem at 1st order

- **in Quantum Mechanics:**

$$\frac{\partial E_\lambda}{\partial \lambda} = \langle \phi_\lambda | \frac{\partial H_\lambda}{\partial \lambda} | \phi_\lambda \rangle$$

H_λ: perturbed Hamiltonian of the system

E_λ: energy eigenvalue of the perturbed system

ϕ_λ: eigenfunction of the perturbed system

- **expectation value of the perturbation of a system is related to the shift in the energy eigenvalue**

- **in Lattice QCD: energy shifts in the presence of a weak external field**

$$S \rightarrow S(\lambda) = S + \lambda \int d^4x\, \mathcal{O}(x) \longrightarrow \bar{q}(x)\Gamma_\mu q(x) \quad , \Gamma_\mu \in \{\mathbf{1}, \gamma_\mu, \gamma_5\gamma_\mu, \ldots\}$$

e.g. local bilinear operator

real parameter

@ 1st order

$$\frac{\partial E_\lambda}{\partial \lambda} = \frac{1}{2E_\lambda} \langle 0 | \mathcal{O} | 0 \rangle$$

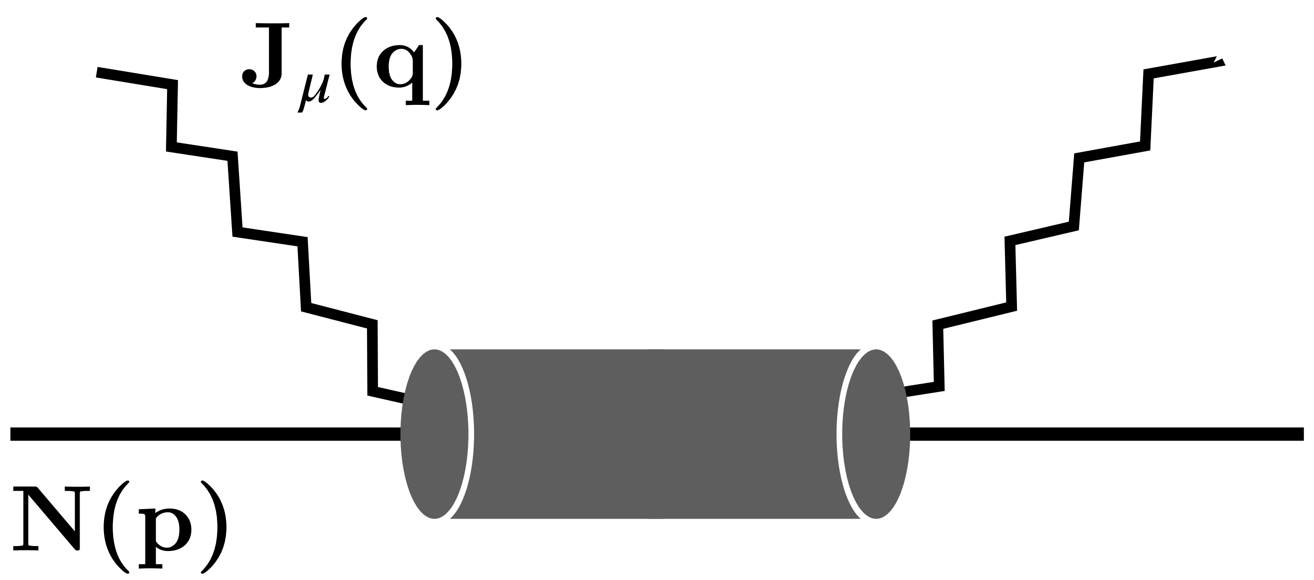
$\boxed{E_\lambda \rightarrow \textbf{spectroscopy, 2-pt function}}$

$\boxed{\langle 0 | \mathcal{O} | 0 \rangle \rightarrow \textbf{determine 3-pt}}$

Applications:

- σ **- terms**
- **Form factors**

Compton Amplitude from FHT at 2nd order



- **unpolarised Compton Amplitude**

$$T_{\mu\mu}(p,q) = \int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p) \,|\, \mathcal{T}\{J_\mu(z) J_\mu(0)\} \,|\, N(p) \rangle$$

4-pt function

- **Action modification**

$$S \to S(\lambda) = S + \lambda \int d^4z \, (e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}}) \, J_\mu(z)$$

local EM current

$$J_\mu(z) = \sum_q e_q \bar{q}(z) \gamma_\mu q(z)$$

$$\left. \frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial \lambda^2} \right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p) \,|\, \mathcal{T}\{J_\mu(z) J_\mu(0)\} \,|\, N(p) \rangle}^{T_{\mu\mu}(p,q)} + q \to -q$$

Determine the Compton Amplitude from second order energy shifts!

Compton Amplitude from FHT at 2nd order

- Spectral decomposition of a 2-point nucleon correlator in an external field, Ω_λ,

$$G_\lambda^{(2)}(\mathbf{p};t) \equiv \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \mathbf{\Gamma} \langle \Omega_\lambda | \chi(\mathbf{x},t)\bar{\chi}(0) | \Omega_\lambda \rangle \simeq A_\lambda(\mathbf{p}) e^{-E_{N_\lambda}(\mathbf{p})t}$$

- Take the 2nd order derivative,

Non-Breit frame, $|\mathbf{p}| \neq |\mathbf{p} \pm \mathbf{q}| \Rightarrow 0$

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = e^{-E_N(\mathbf{p})t}\left[\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial\lambda^2} - t\left(2\frac{\partial A_\lambda(\mathbf{p})}{\partial\lambda}\cancel{\frac{\partial E_{N_\lambda}(\mathbf{p})}{\partial\lambda}} + A(\mathbf{p})\frac{\partial^2 E_{N_\lambda}}{\partial\lambda^2}\right) + t^2 A(\mathbf{p})\cancel{\left(\frac{\partial E_{N_\lambda}(\mathbf{p})}{\partial\lambda}\right)^2}\right]$$

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \left(\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial\lambda^2} - tA(\mathbf{p})\boxed{\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}}\right) e^{-E_N(\mathbf{p})t}$$

quadratic energy shift

temporal enhancement $\sim t\, e^{-E_N(\mathbf{p})t}$

Compton Amplitude from FHT at 2<u>nd</u> order

- **2-point nucleon correlator <u>in path integral formalism</u>,**

$${}_\lambda\langle\chi(\mathbf{x},t)\bar{\chi}(0)\rangle_\lambda = \frac{1}{\mathcal{Z}(\lambda)}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}U\chi(\mathbf{x},t)\bar{\chi}(0)e^{-S(\lambda)}$$

, where

$$S(\lambda) = S + \lambda\int d^4z\left(e^{iq\cdot z} + e^{-iq\cdot z}\right)\mathcal{J}_\mu(z)$$

for simplicity define: $\mathcal{G} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\mathbf{\Gamma}\chi(\mathbf{x},t)\bar{\chi}(0)$

- **Take the 2<u>nd</u> order derivative,**

$$\frac{\partial^2\langle\mathcal{G}\rangle_\lambda}{\partial\lambda^2} = \langle\mathcal{G}\rangle_\lambda\left\langle\cancel{\frac{\partial^2 S(\lambda)}{\partial\lambda^2}}\right\rangle_\lambda + \left\langle\mathcal{G}\cancel{\frac{\partial^2 S(\lambda)}{\partial\lambda^2}}\right\rangle_\lambda + \boxed{\langle\mathcal{G}\rangle_\lambda\left\langle\left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2\right\rangle_\lambda} + \cancel{2\langle\mathcal{G}\rangle_\lambda\left\langle\frac{\partial S(\lambda)}{\partial\lambda}\right\rangle_\lambda\left\langle\frac{\partial S(\lambda)}{\partial\lambda}\right\rangle_\lambda} - 2\left\langle\mathcal{G}\frac{\partial S(\lambda)}{\partial\lambda}\right\rangle_\lambda\left\langle\cancel{\frac{\partial S(\lambda)}{\partial\lambda}}\right\rangle_\lambda + \left\langle\mathcal{G}\left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2\right\rangle_\lambda$$

no quadratic perturbation = 0

does not vanish in general, but only affects the free-field correlator

as $\lambda \to 0$, vacuum m.e. of ext. current $\langle\partial S(\lambda)/\partial\lambda\rangle = 0$, given that the operator does not carry vacuum quantum numbers. EM current satisfies this condition.

- **Thus the second order energy shift comes from,**

$$\left.\frac{\partial^2\langle\mathcal{G}\rangle_\lambda}{\partial\lambda^2}\right|_{\lambda=0} = \left\langle\mathcal{G}\left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2\right\rangle + \ldots$$

terms that are <u>not</u> time enhanced

Compton Amplitude from FHT at $2^{\underline{\text{nd}}}$ order

- **back to full form,**

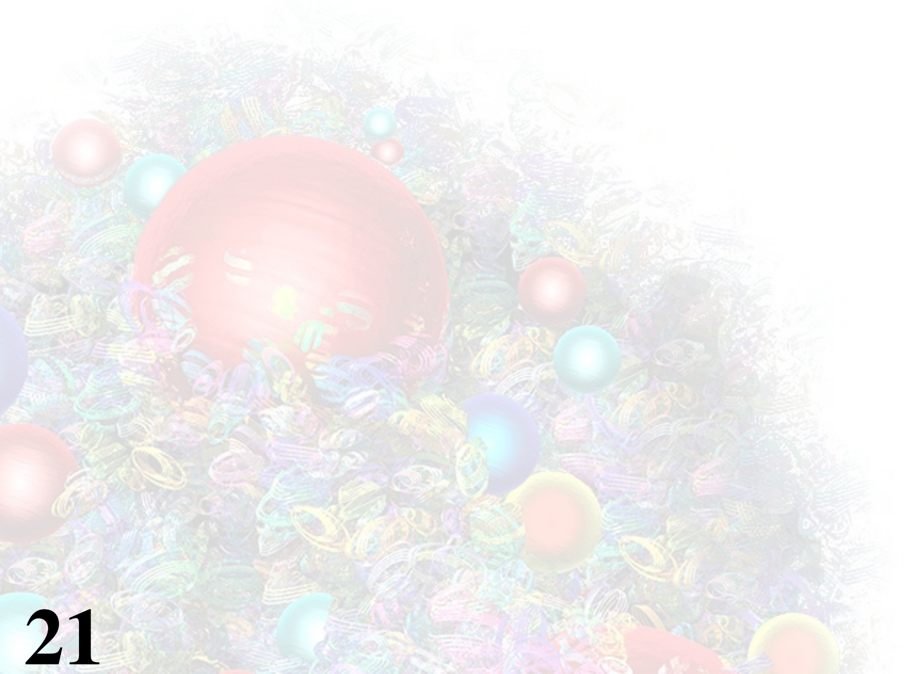
$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};y)}{\partial\lambda^2}\right|_{\lambda=0} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\boldsymbol{\Gamma}\left\langle \chi(\mathbf{x},t)\bar{\chi}(0)\left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2\right\rangle, \text{ where } \frac{\partial S(\lambda)}{\partial\lambda} = \int d^4z(e^{iq\cdot z}+e^{-iq\cdot z})\mathcal{J}_\mu(z)$$

note that $\langle\cdots\rangle$ is evaluated in the absence of the external field

- **writing the $2^{\underline{\text{nd}}}$ order derivative explicitly,**

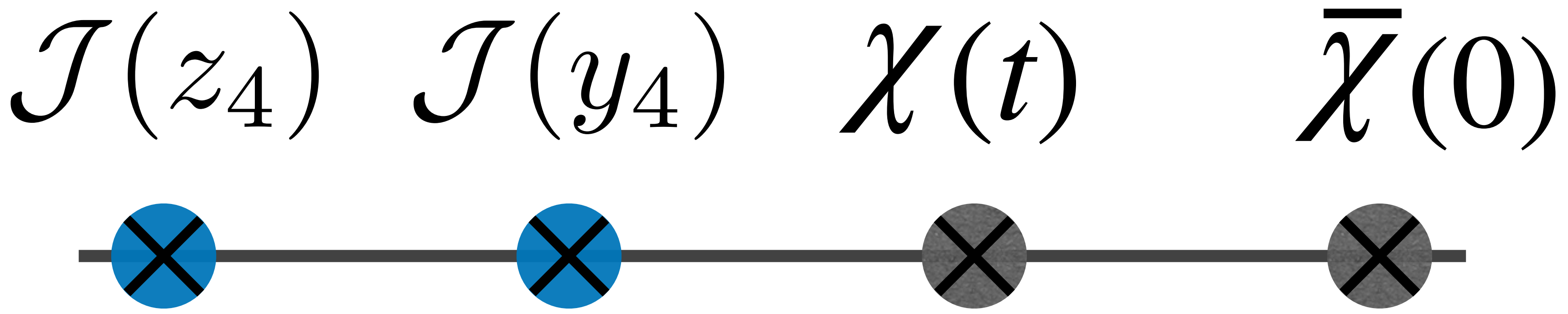
$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\boldsymbol{\Gamma}\int d^4y d^4z(e^{i\mathbf{q}\cdot\mathbf{y}}+e^{-i\mathbf{q}\cdot\mathbf{y}})(e^{i\mathbf{q}\cdot\mathbf{z}}+e^{-i\mathbf{q}\cdot\mathbf{z}})\langle\chi(\mathbf{x},t)\mathcal{J}_\mu(z)\mathcal{J}_\mu(y)\bar{\chi}(0)\rangle$$

<u>need to resolve the time ordering of the currents</u>



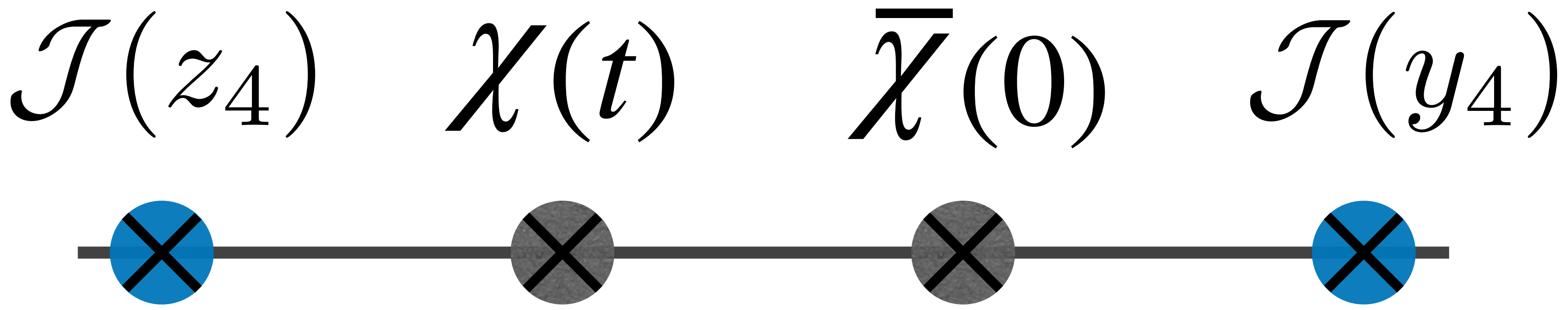
Compton Amplitude from FHT at 2nd order

- **possible time orderings and their contributions:**



$$\sim e^{-E_X t}, \quad E_X \gtrsim E_N$$

no time enhancement



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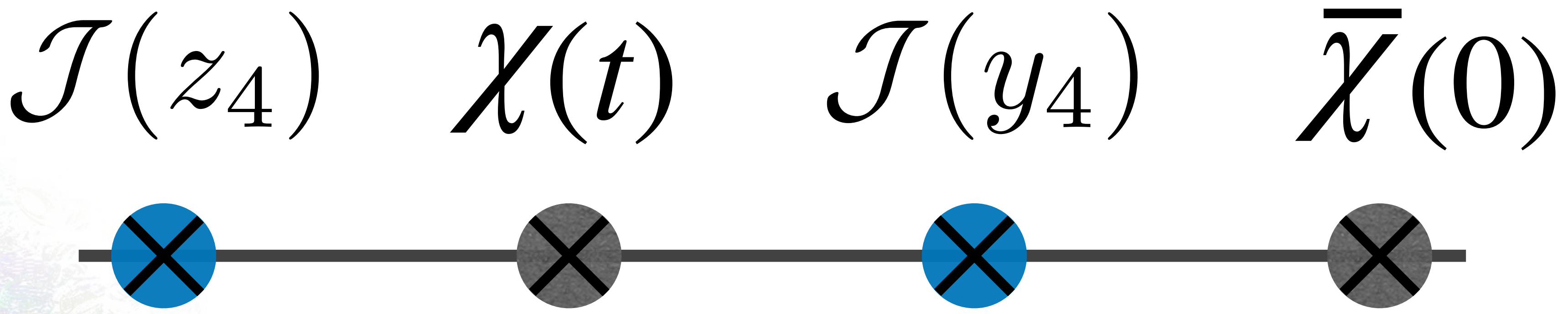
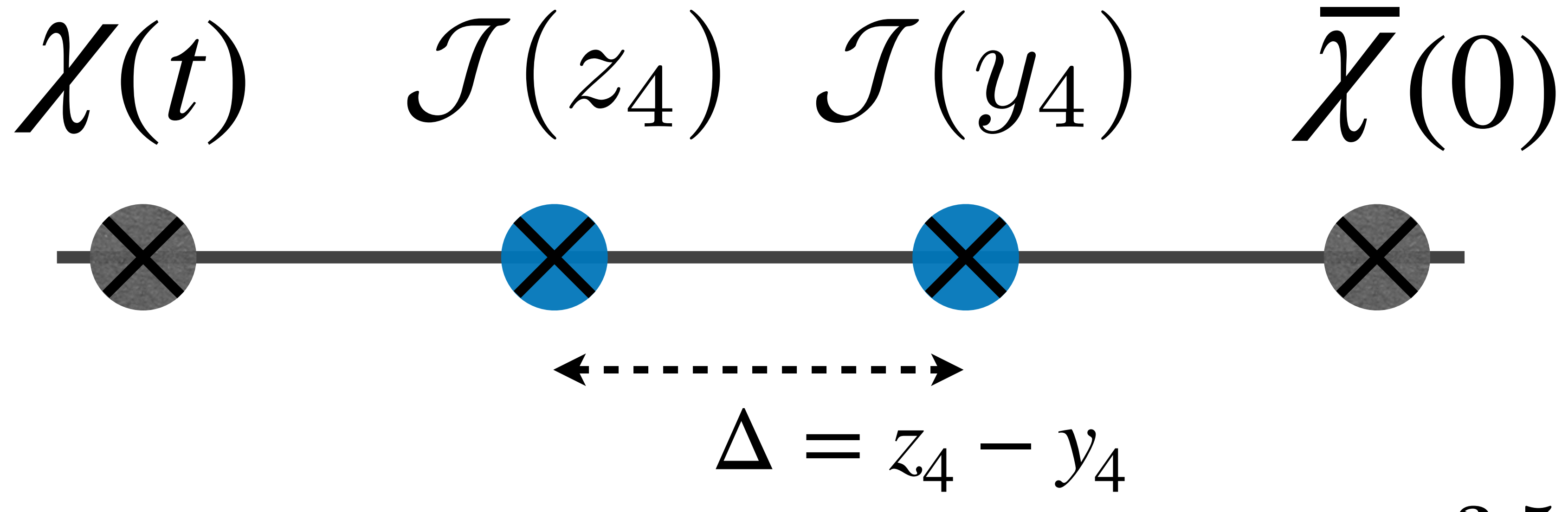


$$\sim t\, e^{-E_N t} \frac{\partial E_N}{\partial \lambda} \to 0$$

there is time enhancement,
but due to non-Breit frame kinematics → 0

Compton Amplitude from FHT at 2nd order

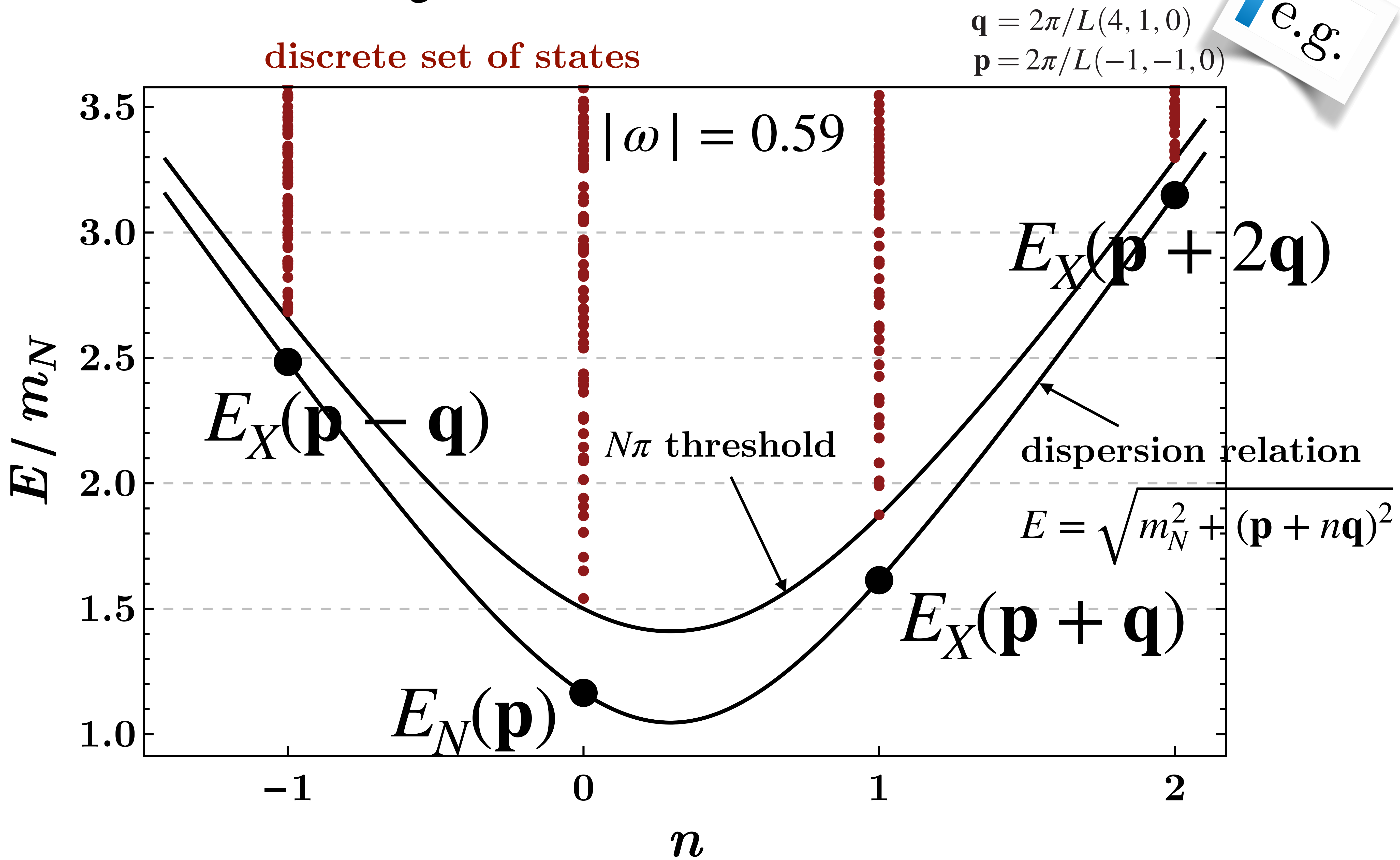
- **relevant contribution comes from the ordering where the currents are sandwiched**



$$\sim t \int^{t} d\Delta\, e^{-\Delta\left(E_X(\mathbf{p}+\mathbf{q}) - E_N(\mathbf{p})\right)}$$

- under the condition $|\omega| < 1$, $E_X(\mathbf{p}+n\mathbf{q}) \gtrsim E_N(\mathbf{p})$, so the intermediate states cannot go on-shell

- ground state dominance is ensured in the large time limit



Compton Amplitude from FHT at 2nd order

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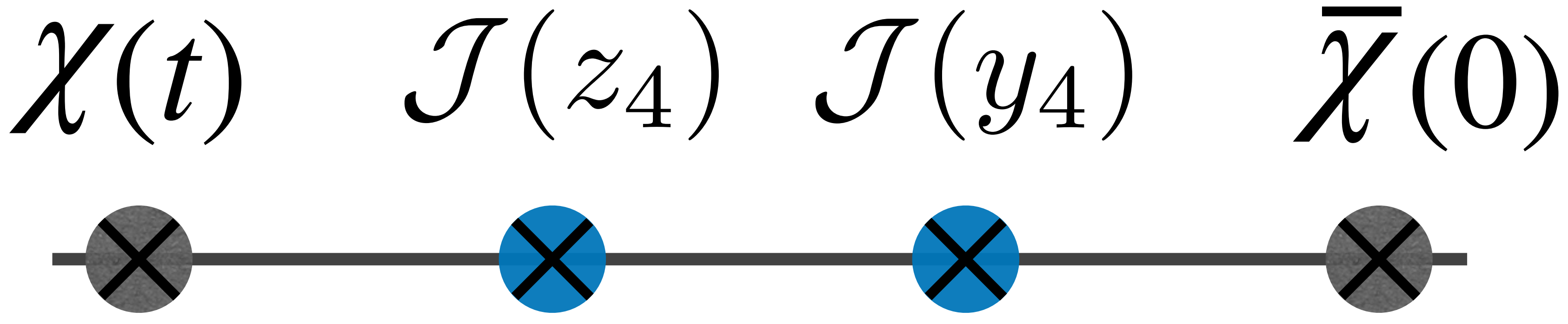


$$\left.\frac{\partial^2 G^{(2)}_\lambda(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = 2\int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \int d^3y d^3z \int_0^t d\tau' \int_0^{\tau'} d\tau (e^{i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{q}\cdot\mathbf{y}})(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}})\mathbf{\Gamma}\langle\chi(x)|\mathcal{J}_\mu(\mathbf{z},\tau')\mathcal{J}_\mu(\mathbf{y},\tau)|\bar{\chi}(0)\rangle$$

$\sum_X |X\rangle\langle X|$ $\sum_Y |Y\rangle\langle Y|$

insert sets of complete states, and use translational invariance,

$$\left.\frac{\partial^2 G^{(2)}_\lambda(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = 2\int d^3y d^3z \int_0^t d\tau' \int_0^{\tau'} d\tau \sum_{X,Y} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-E_X(\mathbf{p})t} e^{-(E_Y(\mathbf{k})-E_X(\mathbf{p}))\tau}}{4E_X(\mathbf{p})E_Y(\mathbf{k})} e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}(e^{i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{q}\cdot\mathbf{y}})(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}})$$
$$\times\,\mathbf{\Gamma}\langle\Omega|\chi(0)|X(\mathbf{p})\rangle\langle X(\mathbf{p})|\mathcal{J}_\mu(\mathbf{z}-\mathbf{y},\tau'-\tau)\mathcal{J}_\mu(\mathbf{0},0)|Y(\mathbf{k})\rangle\langle Y(\mathbf{k})|\bar{\chi}(0)|\Omega\rangle.$$

carrying out the integrals and the remaining algebra,

$$\left.\frac{\partial^2 G^{(2)}_\lambda(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z})\langle N(\mathbf{p})|\mathcal{T}\{\mathcal{J}(z)\mathcal{J}(0)\}|N(\mathbf{p})\rangle$$

Compton Amplitude from FHT at 2nd order

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \left(\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial\lambda^2} - tA(\mathbf{p})\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right) e^{-E_N(\mathbf{p})t}$$

from spectral decomposition

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z})\langle N(\mathbf{p})|\mathcal{T}\{\mathcal{J}(z)\mathcal{J}(0)\}|N(\mathbf{p})\rangle$$

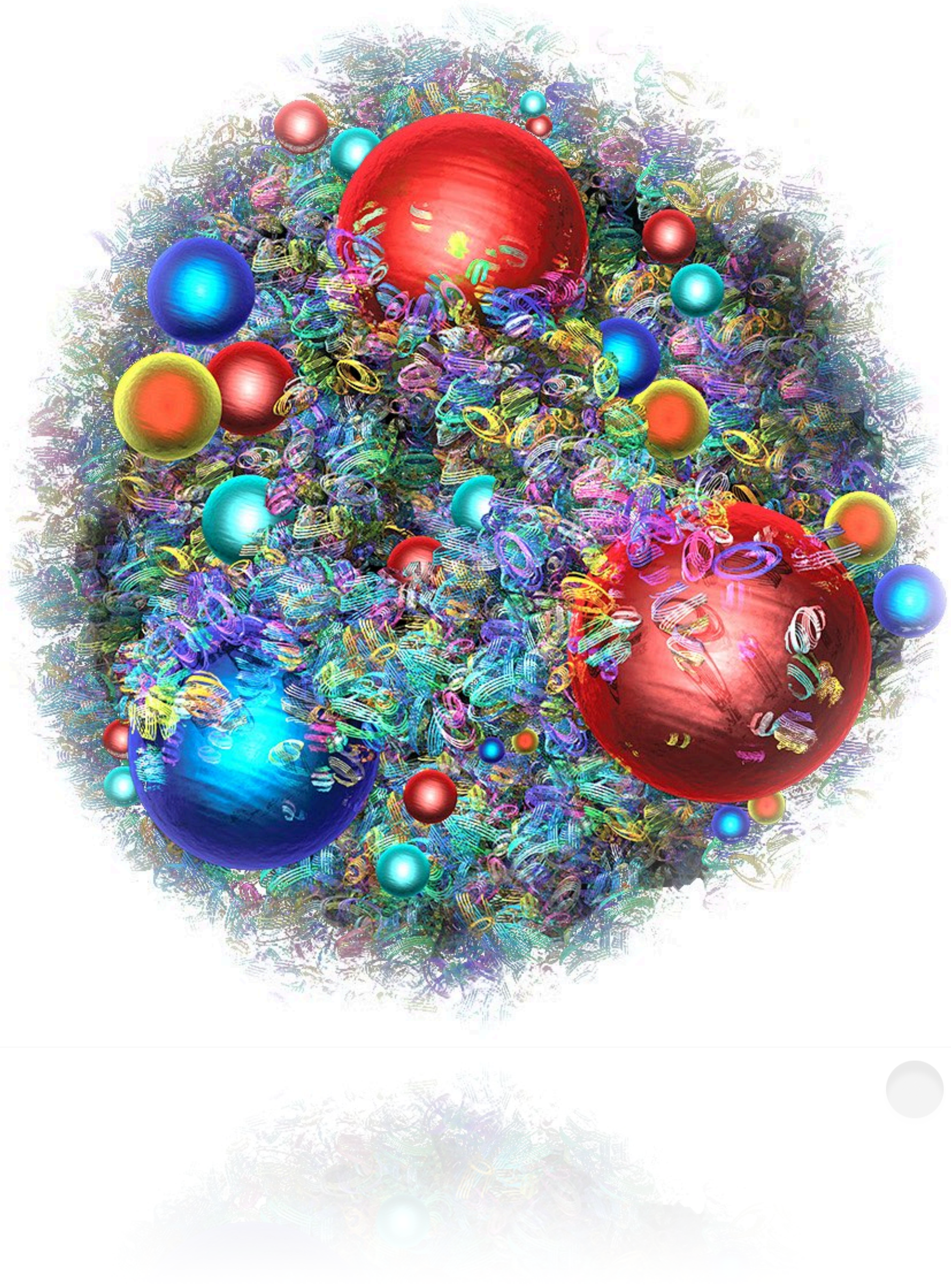
from path integral

- **equate the time-enhanced terms:**

$$\left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z (e^{iq\cdot z} + e^{-iq\cdot z})\langle N(\mathbf{p})|\mathcal{J}(z)\mathcal{J}(0)|N(\mathbf{p})\rangle}^{T_{\mu\mu}(p,q)+T_{\mu\mu}(p,-q)}$$

Compton amplitude is related to the second-order energy shift

Outline

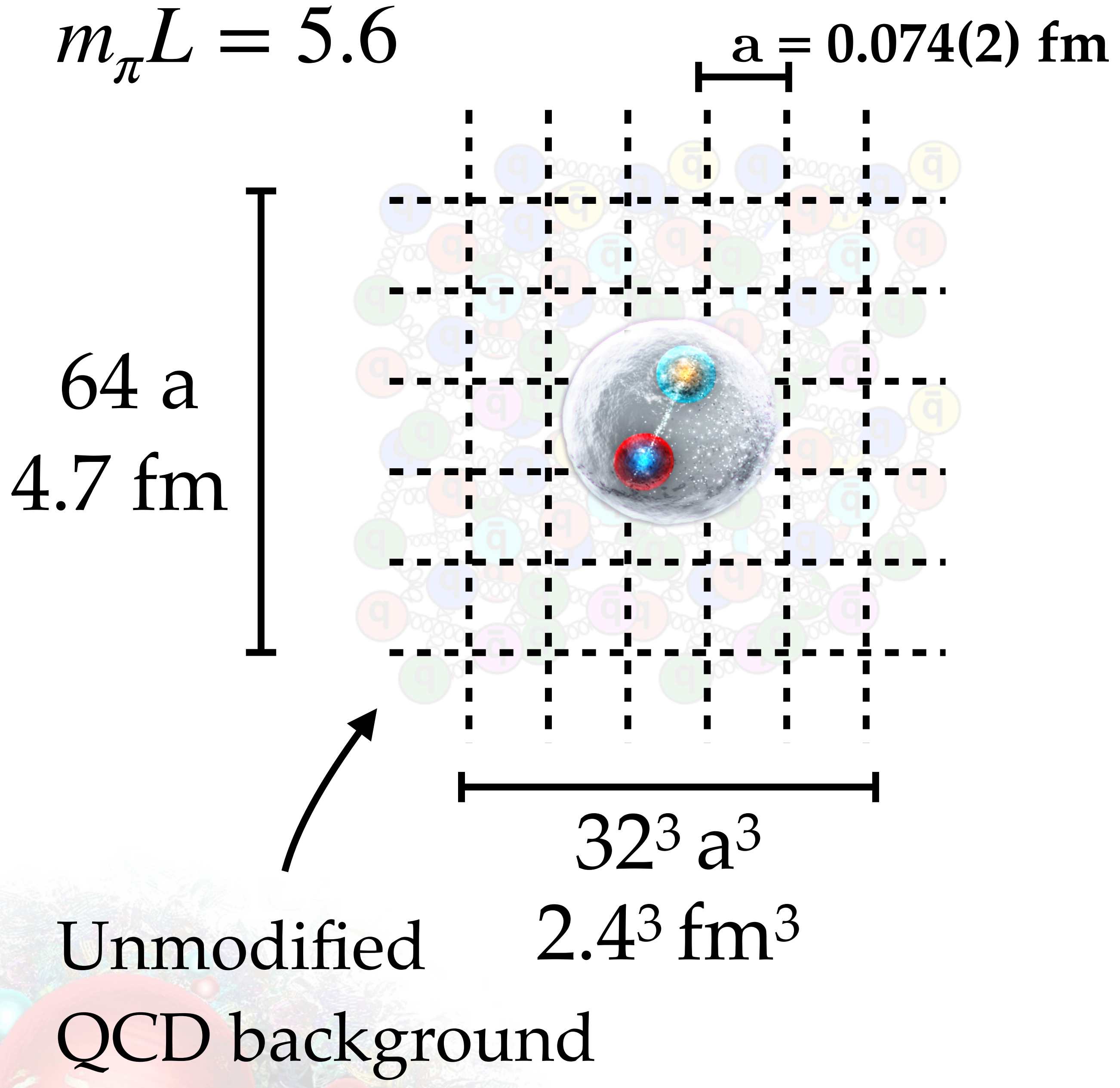


- Forward Compton Amplitude & the Nucleon Structure Functions
- Feynman-Hellmann Theorem & the Compton Amplitude
- Moments of the Nucleon Structure Functions
- Scaling and Power Corrections/Higher-twist effects

Lattice Details

QCDSF/UKQCD, 32^3x64, 2+1 flavor (u/d+s)
$\beta = 5.50$, NP-improved Clover action
Phys. Rev. D 79, 094507 (2009), arXiv:0901.3302 [hep-lat]

$m_\pi \sim 470$ MeV, ~SU(3) sym.
$m_\pi L = 5.6$



a = 0.074(2) fm

64 a
4.7 fm

32^3 a^3
2.4^3 fm^3

Unmodified
QCD background

- Valence u/d quarks with modified action, $S(\lambda)$
  - Local EM current insertion, $J_\mu(x) = Z_V \bar{q}(x)\gamma_\mu q(x)$ with $Z_V = 0.8611(84)$
  - Feynman-Hellmann implementation at the valence quark level
- 4 Distinct field strengths, $\lambda = [\pm 0.0125, \pm 0.025]$
- 5 different current momenta in the range, $3 \lesssim Q^2 \lesssim 7\, GeV^2$
- $\mathcal{O}(10^4)$ measurements for each pair of Q^2 and λ
- Access to a range of ω values for several (p, q) pairs
  - An inversion for each q and λ, varying p is relatively cheap
- Connected 2-pt correlators calculated only, no disconnected
- Jacobi-smeared sources and sinks, rms $r \sim 0.5\,$fm
- Statistics from 200 bootstrap samples

Strategy | Kinematic coverage

- Access to a range of ω values for several (p, q) pairs

e.g.

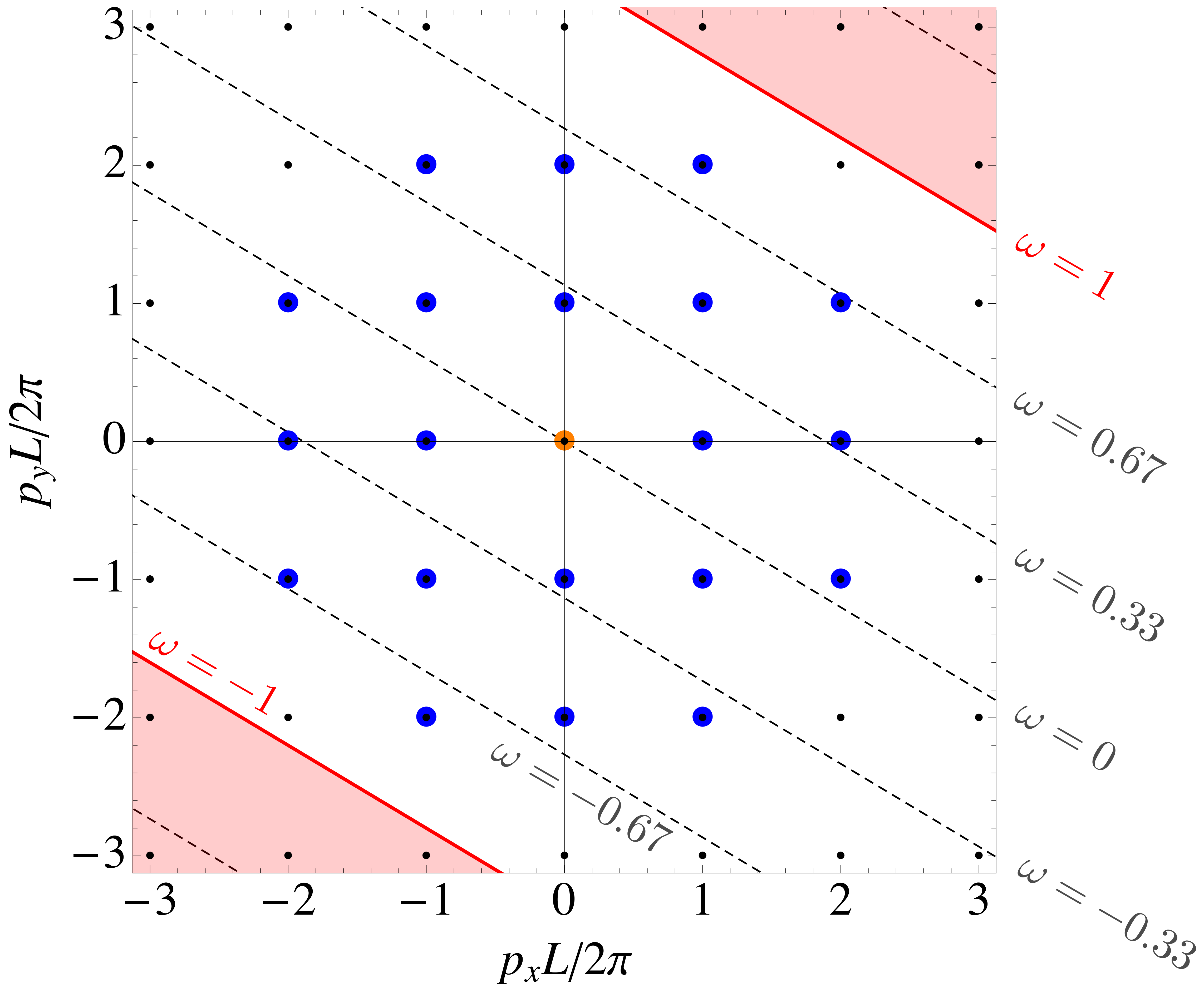
External momentum

$$\vec{q} = (3, 5, 0)\,\frac{2\pi}{L}$$

Can access different ω by varying the nucleon momenta

$$\omega = \frac{2P.q}{Q^2} = \frac{2\vec{P}.\vec{q}}{\vec{q}^2}$$

$q_4 = 0$

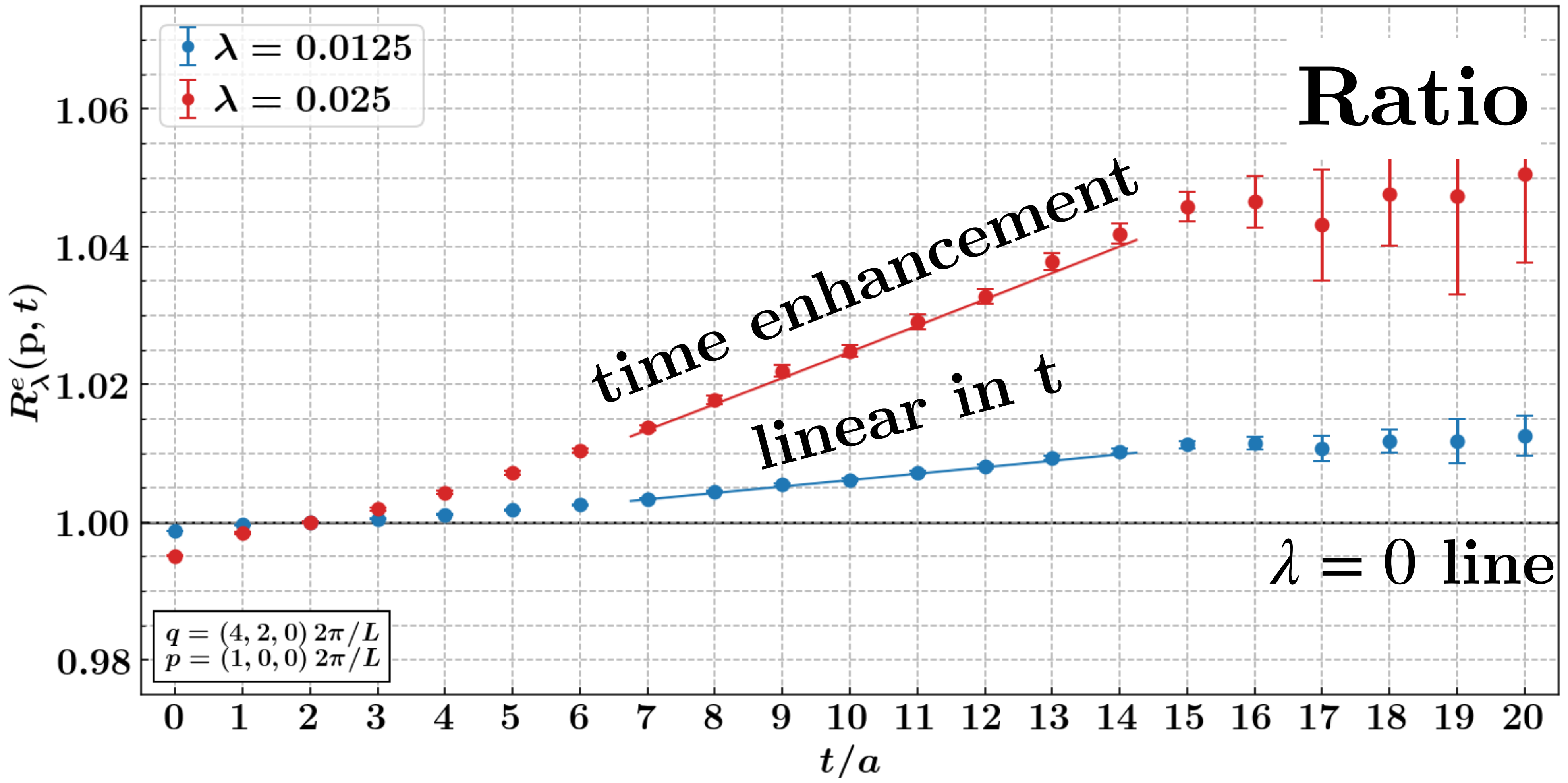


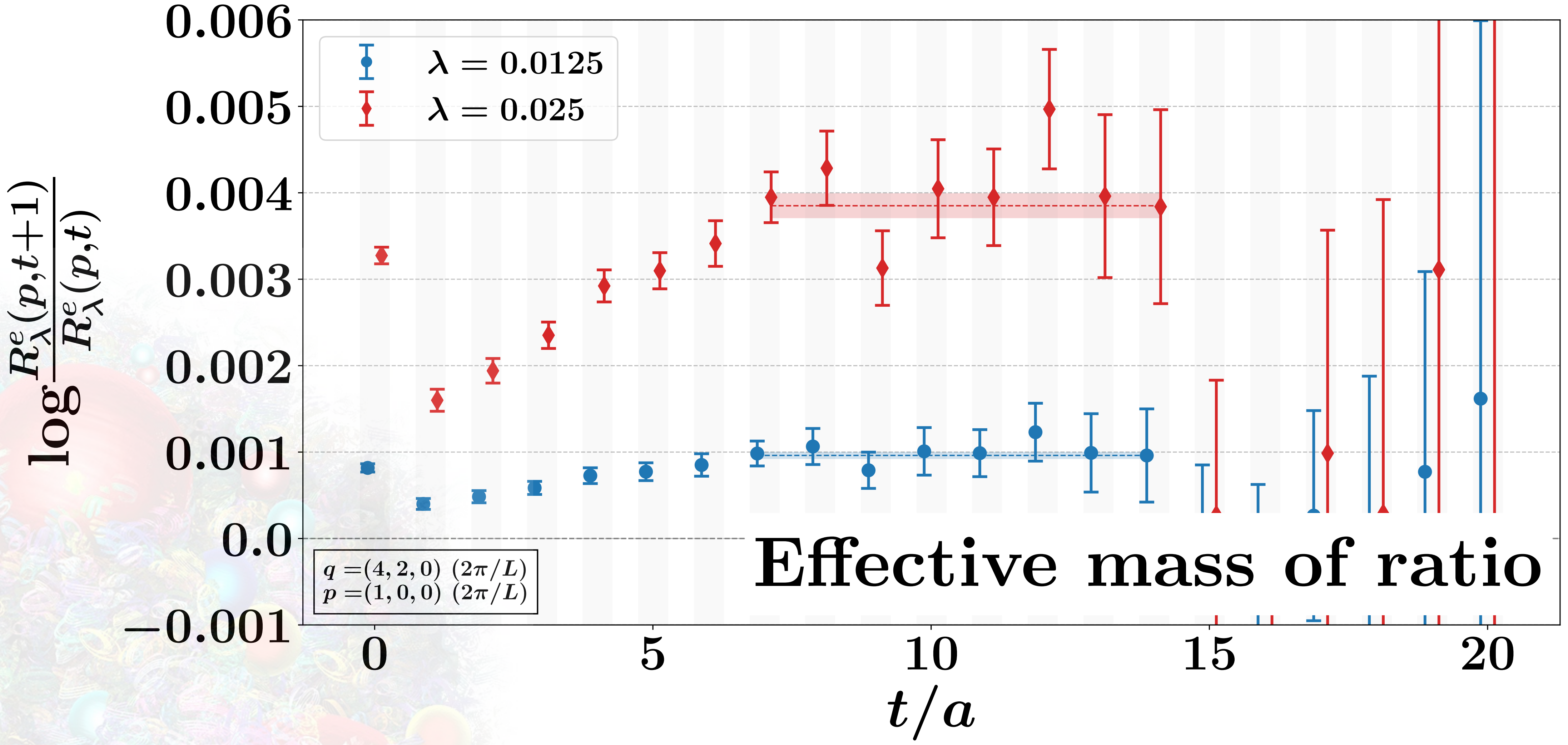
Blue dots: different nucleon Fourier momenta

Strategy | Energy shifts

$a = 0.074\,\text{fm}$
$m_\pi \sim 470\,\text{MeV}$
32^3x64, 2+1 flavour

- Extract energy shifts for each λ





Ratio of perturbed to unperturbed 2-pt functions

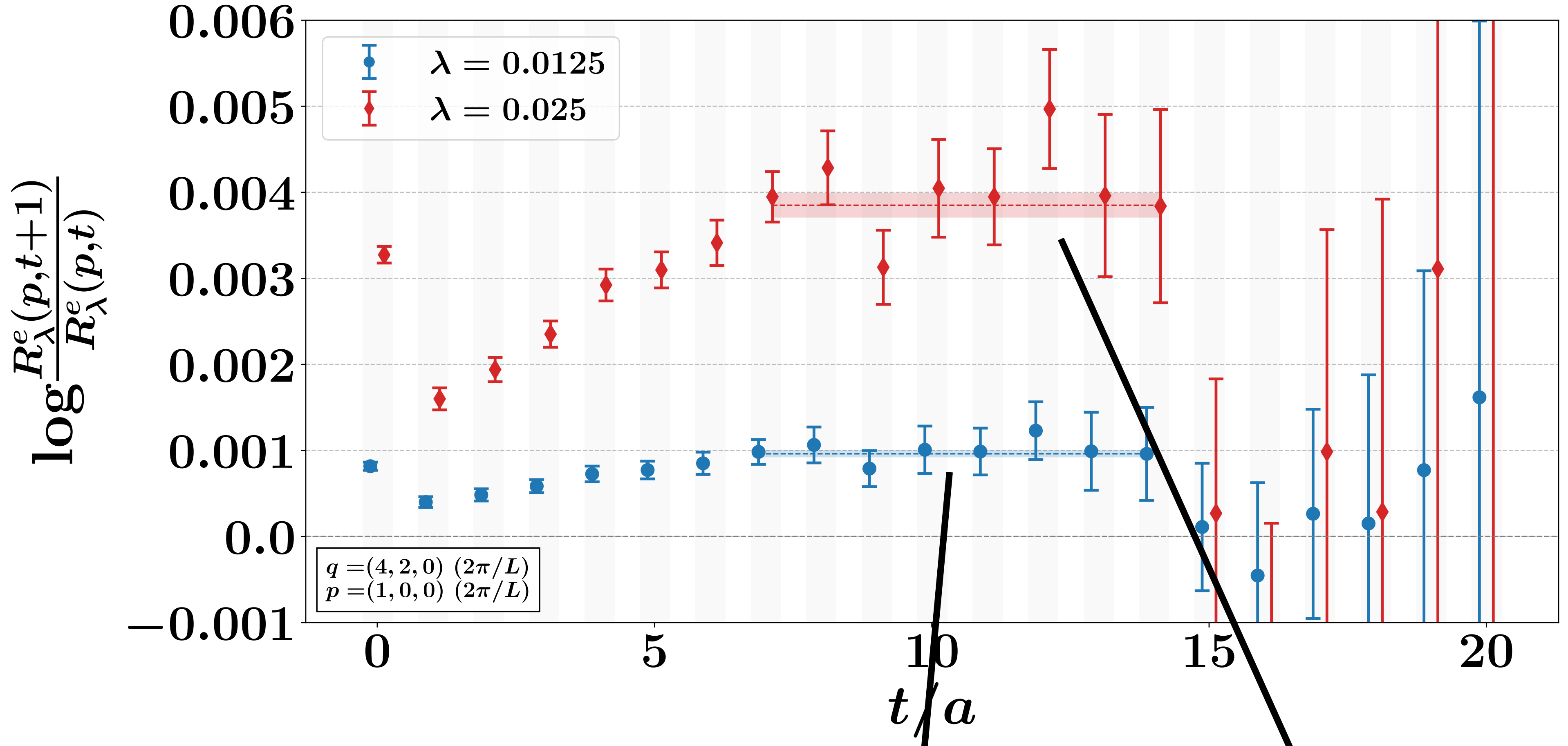
$$R^e_\lambda(\mathbf{p},t) \equiv \frac{G^{(2)}_{+\lambda}(\mathbf{p},t)G^{(2)}_{-\lambda}(\mathbf{p},t)}{\left(G^{(2)}(\mathbf{p},t)\right)^2}$$

$$\xrightarrow{t \gg 0} A_\lambda(\mathbf{p})e^{-2\Delta E^e_{N_\lambda}(\mathbf{p})t}$$

Strategy | Energy shifts

$a = 0.074$ fm
$m_\pi \sim 470$ MeV
32^3x64, 2+1 flavour

- Extract energy shifts for each λ

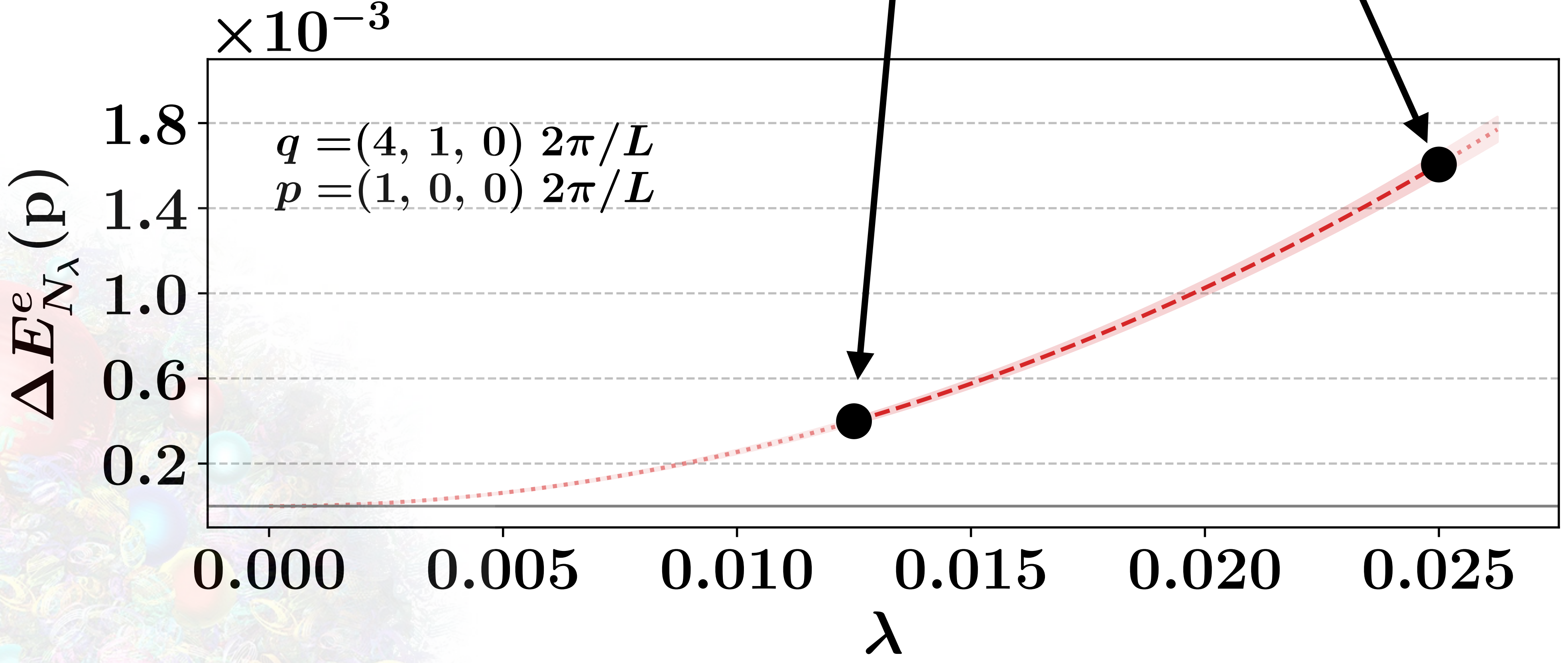


Ratio of perturbed to unperturbed 2-pt functions

$$R^e_\lambda(\mathbf{p},t) \equiv \frac{G^{(2)}_{+\lambda}(\mathbf{p},t)G^{(2)}_{-\lambda}(\mathbf{p},t)}{\left(G^{(2)}(\mathbf{p},t)\right)^2}$$

$$\xrightarrow{t\gg 0} A_\lambda(\mathbf{p})e^{-2\Delta E^e_{N_\lambda}(\mathbf{p})t}$$

- Get the 2nd order derivative



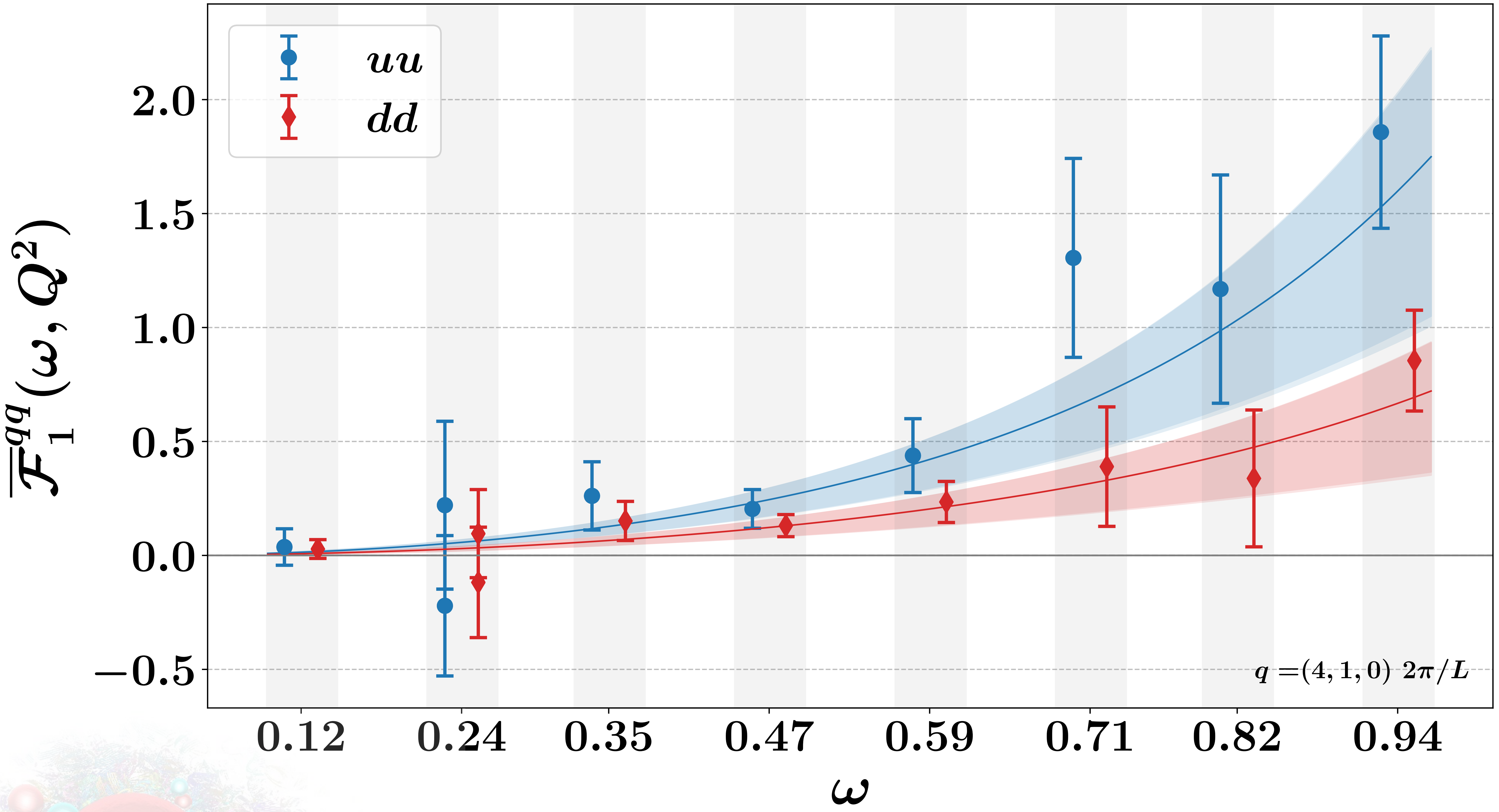
Slope of the curve

$$\Delta E^e_{N_\lambda}(\mathbf{p}) = \frac{\lambda^2}{2}\left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right|_{\lambda=0} + \mathcal{O}(\lambda^4)$$

Strategy | Structure Functions

$a = 0.074\,\mathrm{fm}$
$m_\pi \sim 470\,\mathrm{MeV}$
32^3x64, 2+1 flavour

$$\mathbf{q} = (4,1,0)\,2\pi/L,\ Q^2 = 4.66\,GeV^2$$



Remember our kinematic choices

$\mu = \nu = 3$ and $p_z = q_z = 0$

$$T_{33}(p,q) = \mathcal{F}_1(\omega, Q^2)$$

$$\left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial^2 \lambda}\right|_{\lambda=0} = -\frac{T_{33}(p,q) + T_{33}(p,-q)}{2E_N(\mathbf{p})}$$

$$= -\frac{\mathcal{F}_1(\omega, Q^2)}{E_N(\mathbf{p})}$$

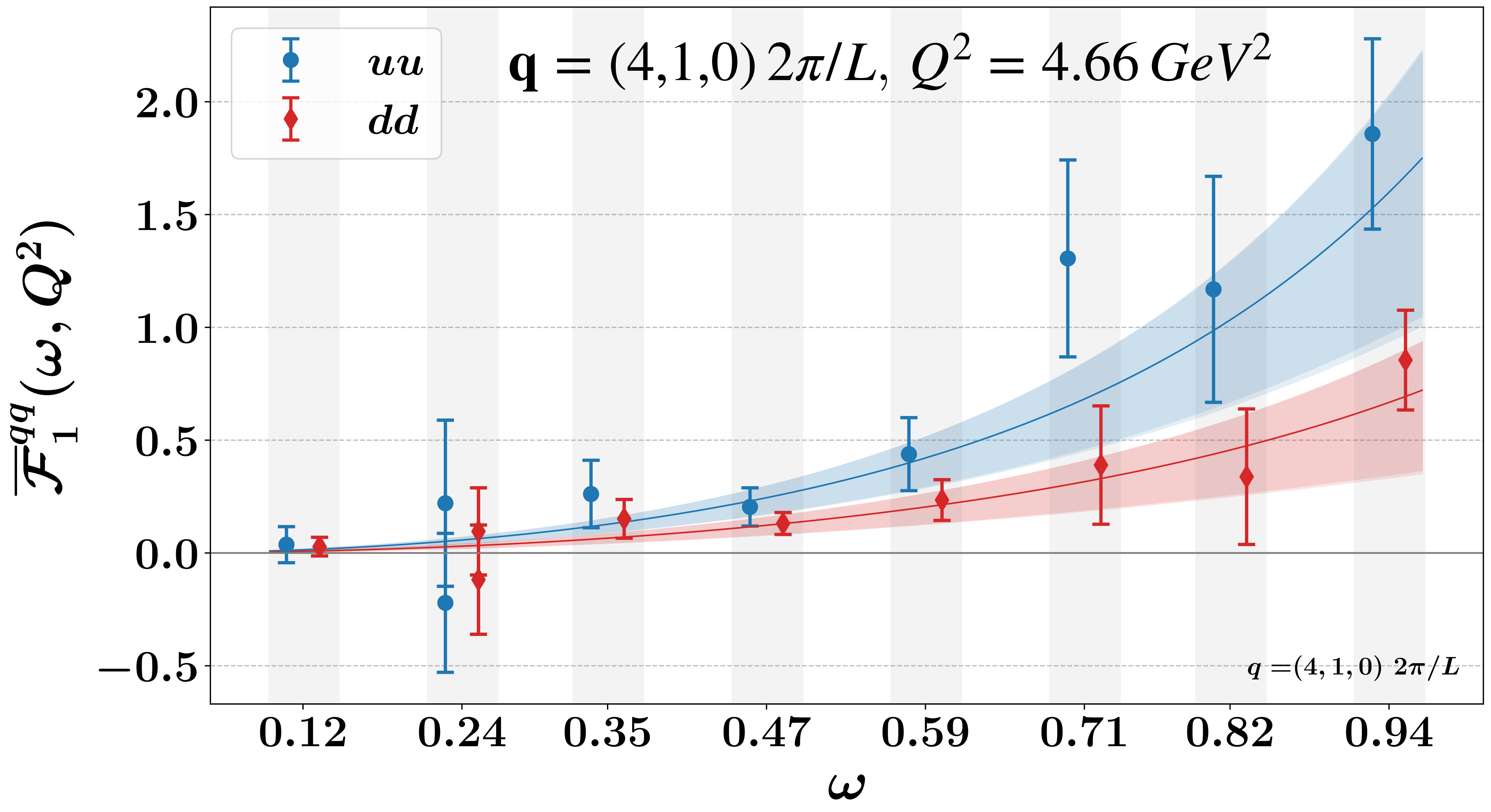
fixed **q** varying **p** → range of ω values

$$\omega = 1/x = \frac{2p\cdot q}{Q^2}$$

Moments | Fit

$a = 0.074\,\mathrm{fm}$
$m_\pi \sim 470\,\mathrm{MeV}$
32^3x64, 2+1 flavour

Remember:
$T_{33}(p,q) = \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$
$T_{33}(p,q) = \mathcal{F}_1(\omega, Q^2)$



$$\overline{\mathcal{F}}_1(\omega, Q^2) = 4(\omega^2 M_2^{(1)}(Q^2) + \omega^4 M_4^{(1)}(Q^2) + \cdots + \omega^{2n} M_{2n}^{(1)}(Q^2) + \cdots)$$

- **Enforce monotonic decreasing of moments for u and d only, not necessarily true for $u-d$**

$$M_2^{(1)}(Q^2) \geq M_4^{(1)}(Q^2) \geq \cdots \geq M_{2n}^{(1)}(Q^2) \geq \cdots \geq 0$$

We truncate at $n = 6$
No dependence to truncation order for $3 \leq n \leq 10$

- **Bayesian approach by MCMC method** — least-squares fluctuates,
tricky to impose monotonic deceasing and positivity bound

Sample the moments from Uniform priors ***individually for u- and d-quark***

$$M_2^{(1)}(Q^2) \sim \mathcal{U}\,(0,1)$$
$$M_{2n}^{(1)}(Q^2) \sim \mathcal{U}\left(0, M_{2n-2}^{(1)}(Q^2)\right)$$

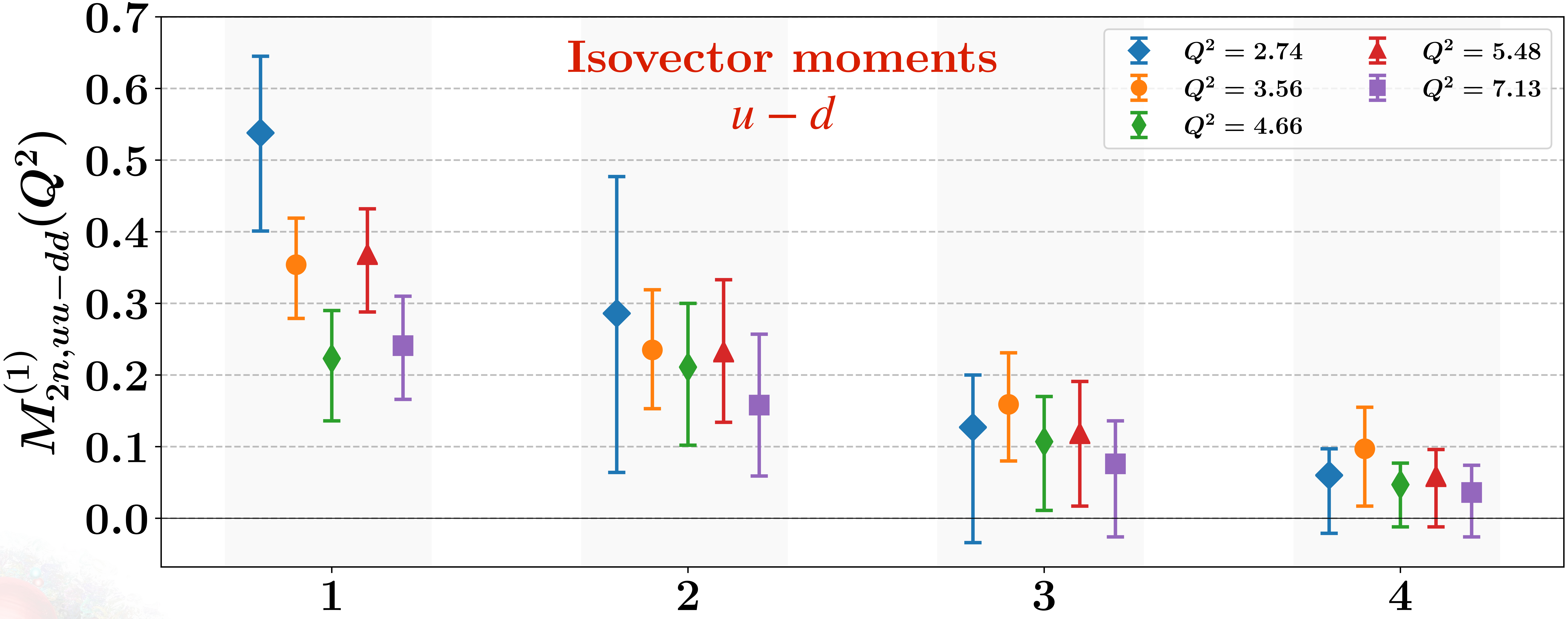
Multivariate Likelihood function, $exp(-\chi^2/2)$

$$\chi^2 = \sum_{i,j} \left[\overline{\mathcal{F}}_{1,i} - \overline{\mathcal{F}}_1^{obs}(\omega_i)\right] \underset{\nearrow}{C_{ij}^{-1}} \left[\overline{\mathcal{F}}_{1,j} - \overline{\mathcal{F}}_1^{obs}(\omega_j)\right]$$

covariance matrix

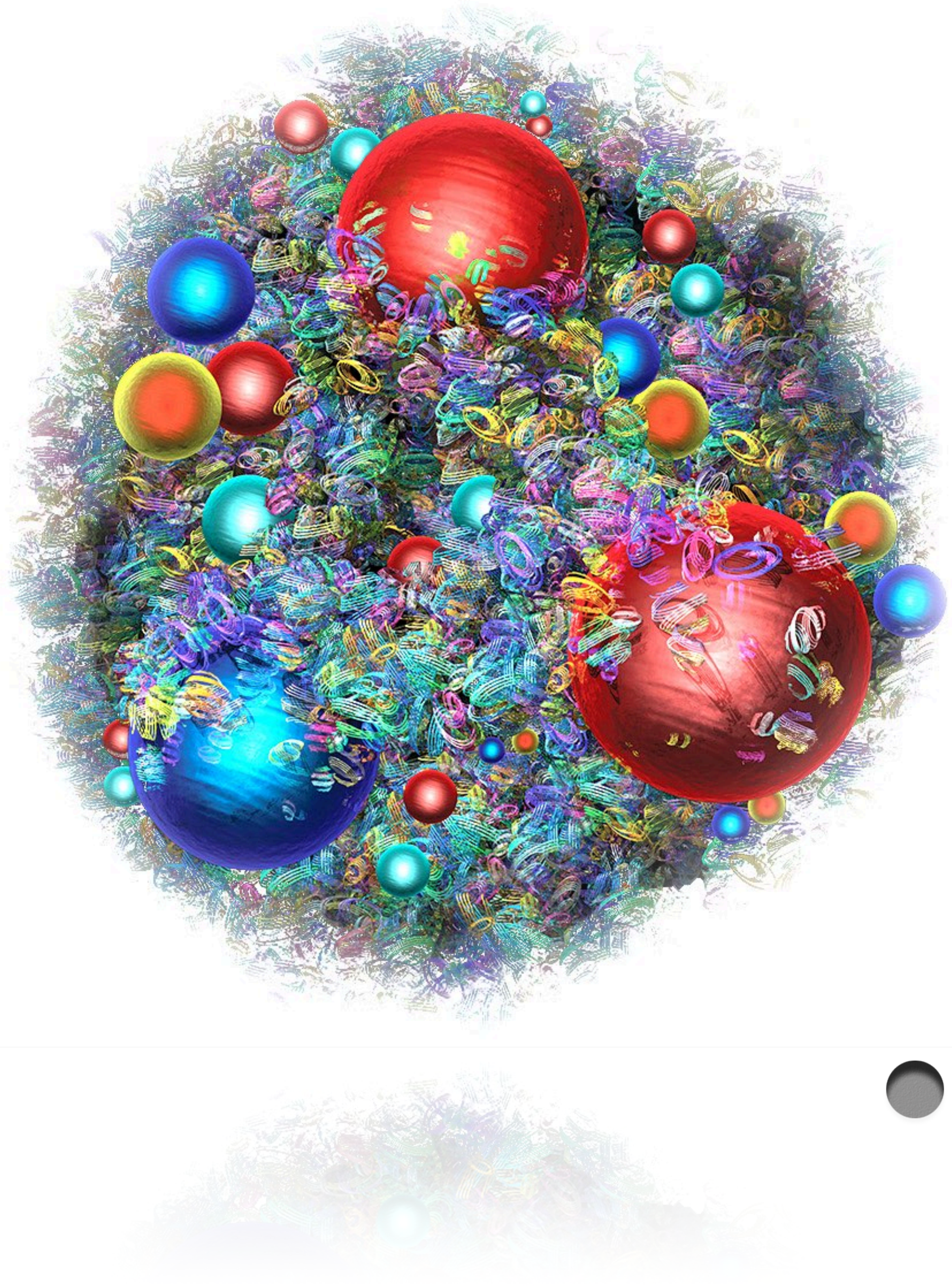
Moments

$a = 0.074\,\mathrm{fm}$
$m_\pi \sim 470\,\mathrm{MeV}$
32^3x64, 2+1 flavour



lowest moment

Outline

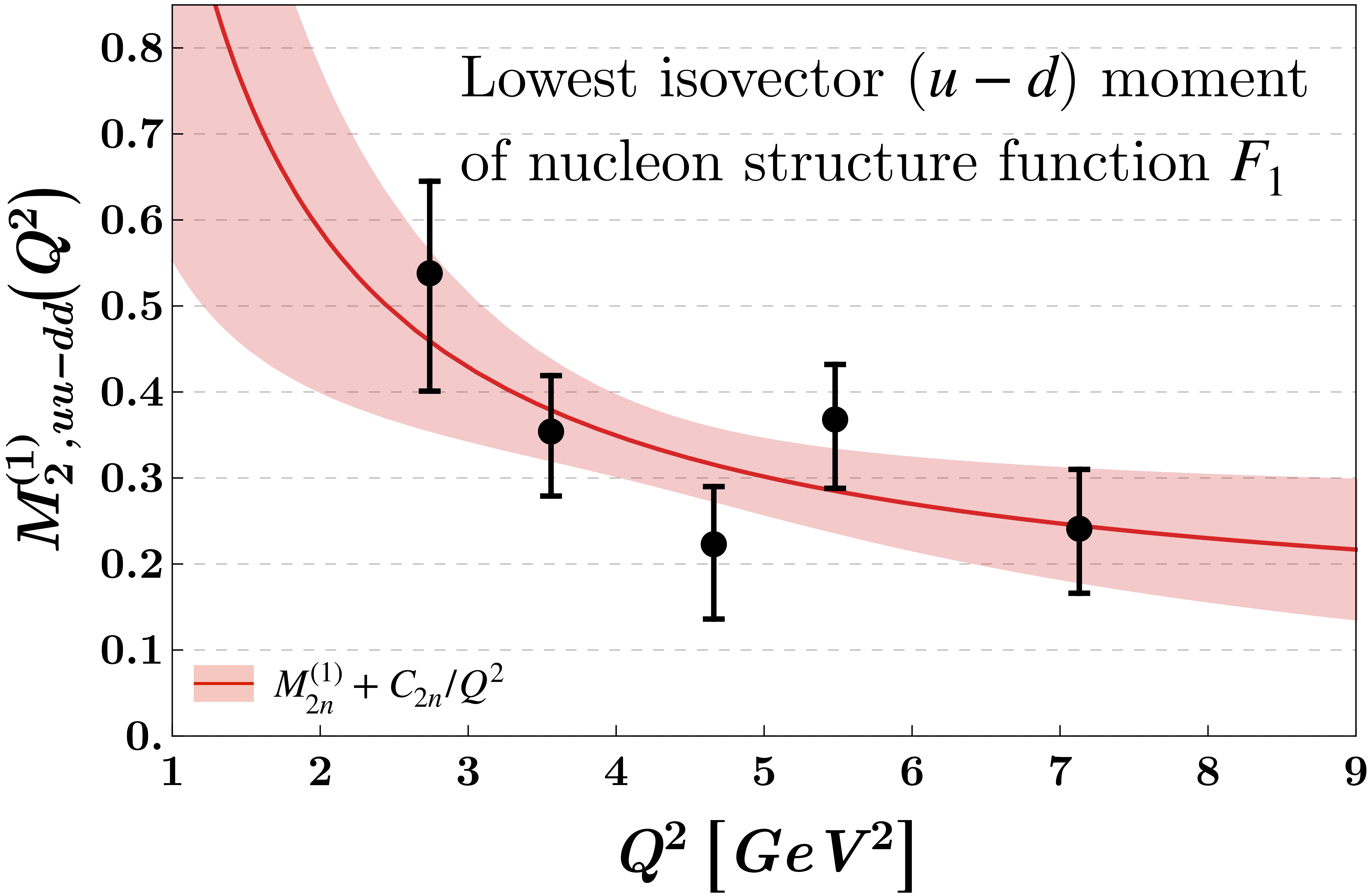


- Forward Compton Amplitude & the Nucleon Structure Functions
- Feynman-Hellmann Theorem & the Compton Amplitude
- Moments of the Nucleon Structure Functions
- Scaling and Power Corrections/Higher-twist effects

Scaling

$a = 0.074\,\text{fm}$
$m_\pi \sim 470\,\text{MeV}$
32^3x64, 2+1 flavour

- **Unique ability to study the Q^2 dependence of the moments!**



Possible for the first time in a lattice simulation!

- Global PDF-fit cuts $\sim 10\,GeV^2$
- Credible scaling region $\sim 16\,GeV^2$
- Need $Q^2 > 10\,GeV^2$ data to reliably extract moments and report at $\mu = 2\,GeV$

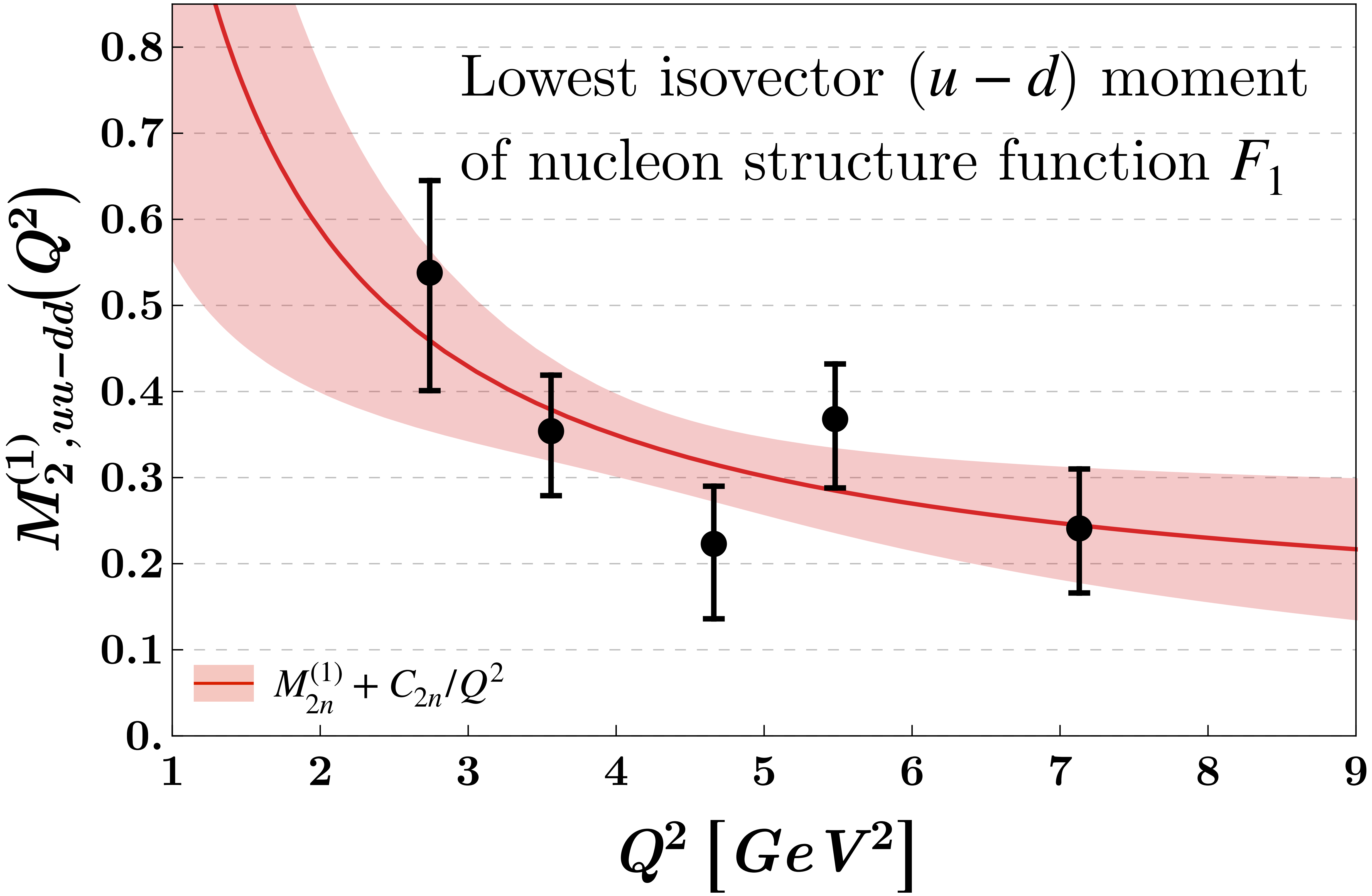
Power corrections

Scaling

Power Corrections

$a = 0.074\,\mathrm{fm}$
$m_\pi \sim 470\,\mathrm{MeV}$
32^3x64, 2+1 flavour

- **Compton amplitude includes all possible power corrections!**

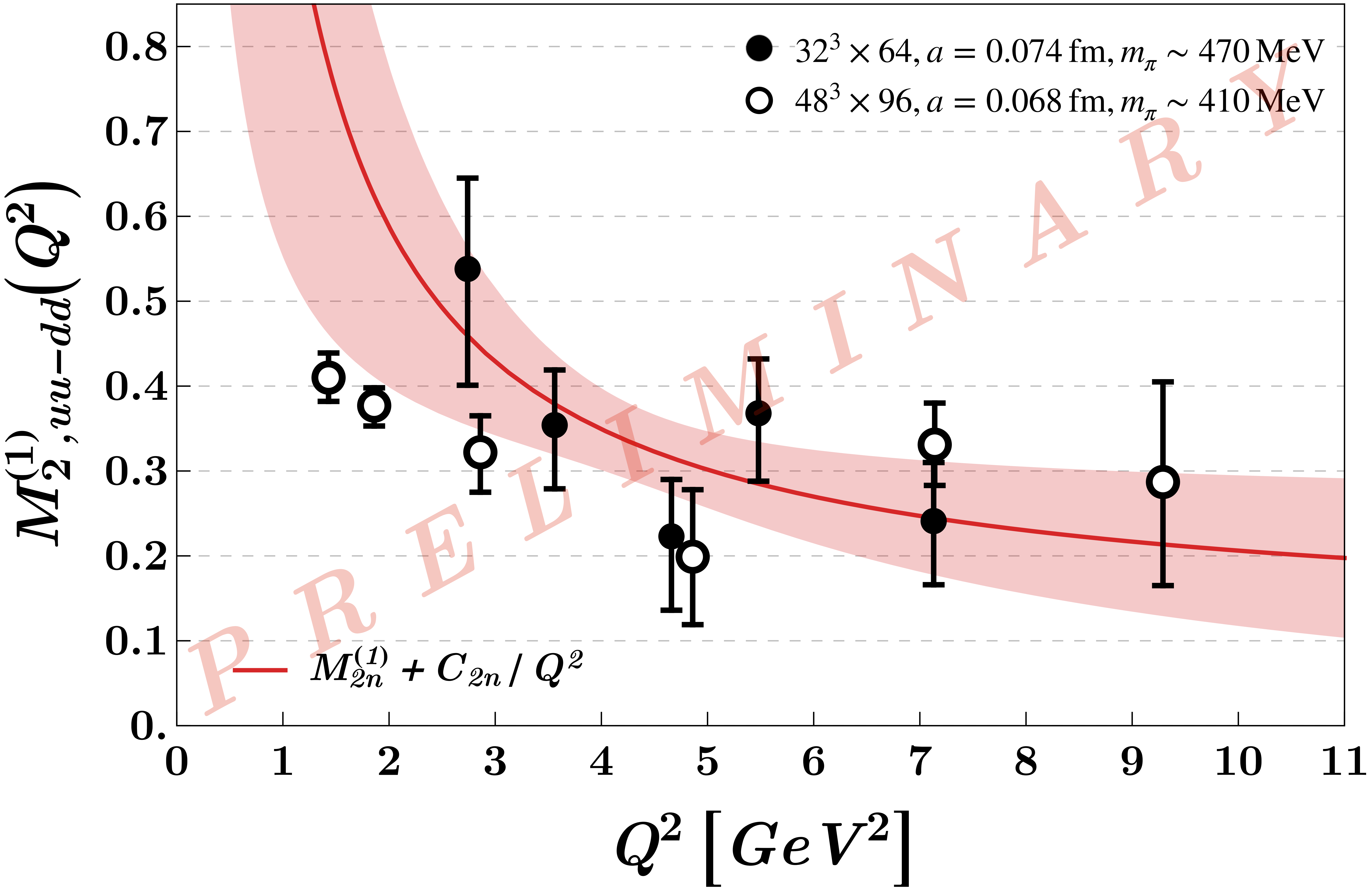


- Power corrections below $\sim 3\,GeV^2$?
  - naïve modelling via
  - $M^{(1)}_{2n}(Q^2) = M^{(1)}_{2n} + C_{2n}/Q^2$
- Need more statistics and lower Q^2 data

Outlook

More on Scaling & Power Corr.

- **Preliminary data points from $48^3 \times 96$ configurations**



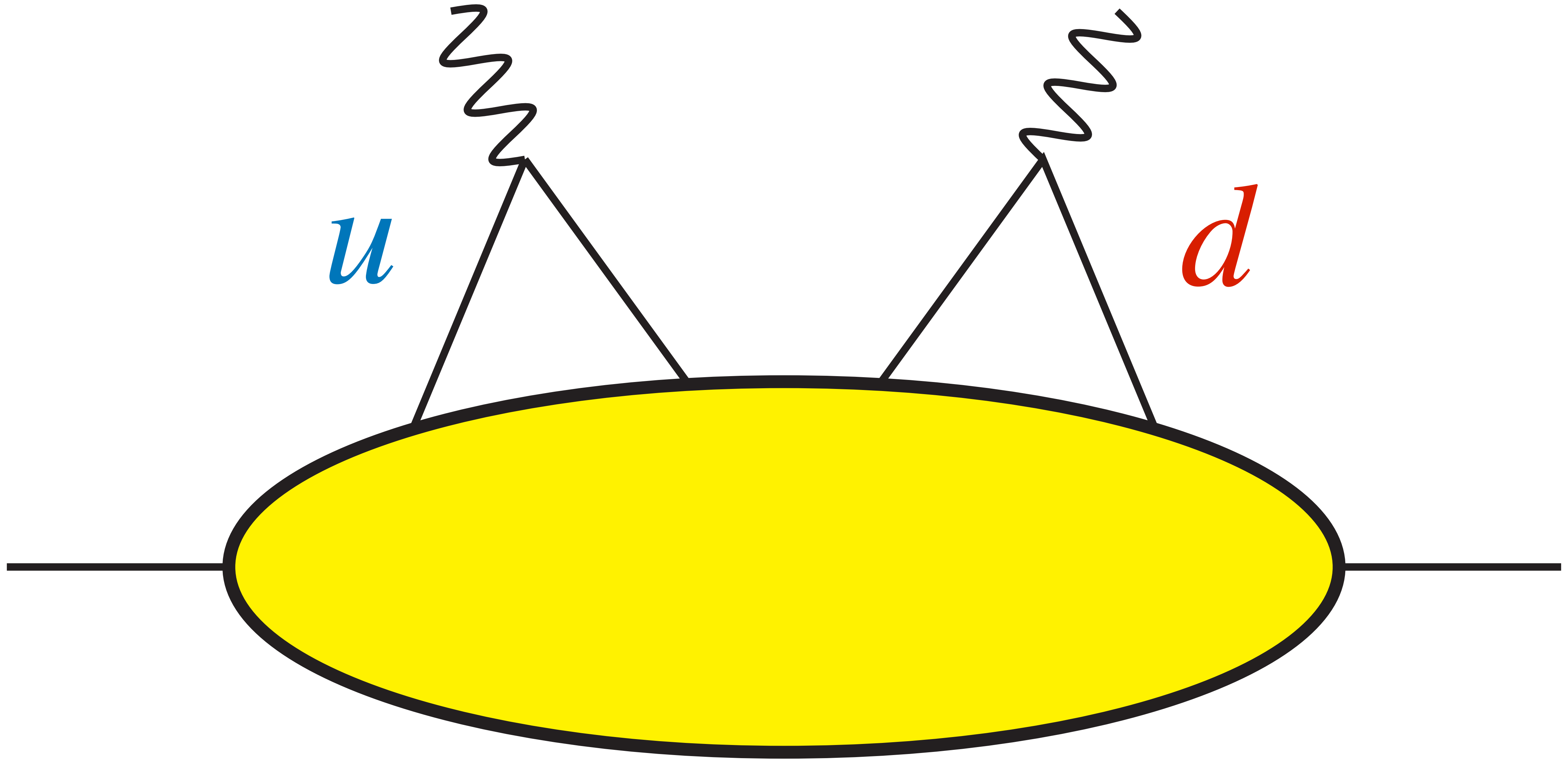
qualitative comparison

no systematics yet

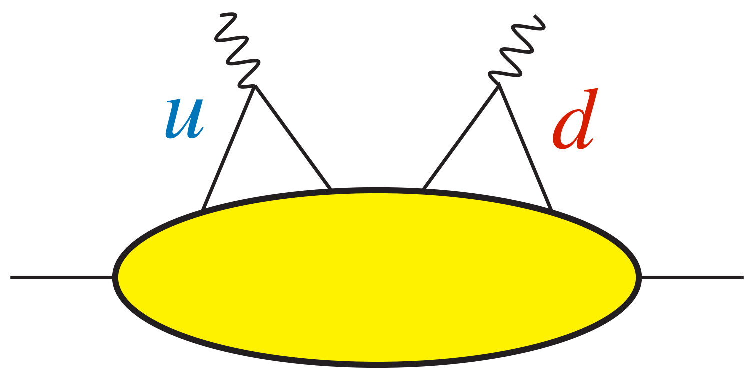
Higher Twist

pure Twist-4 contributions

ud interference term

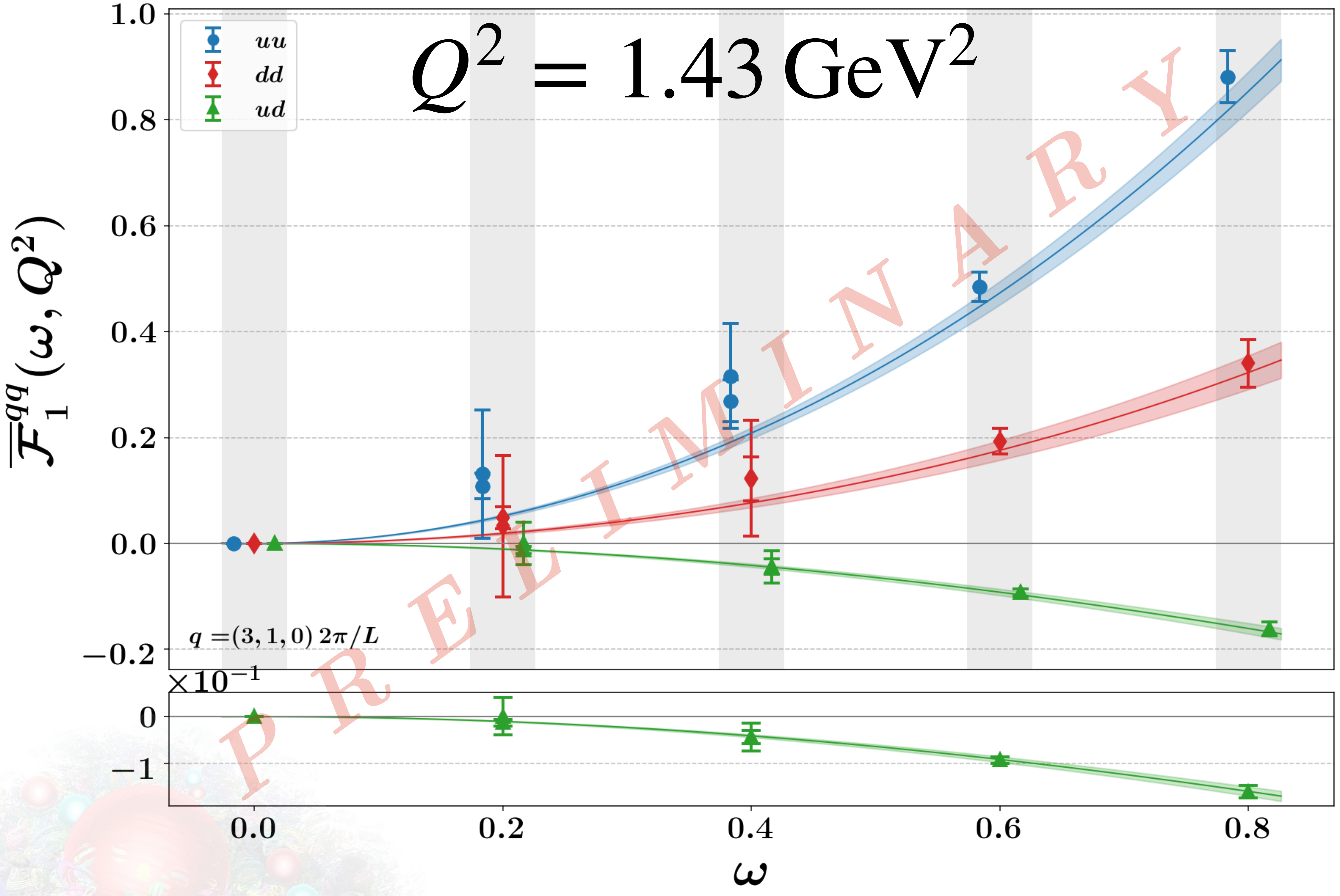


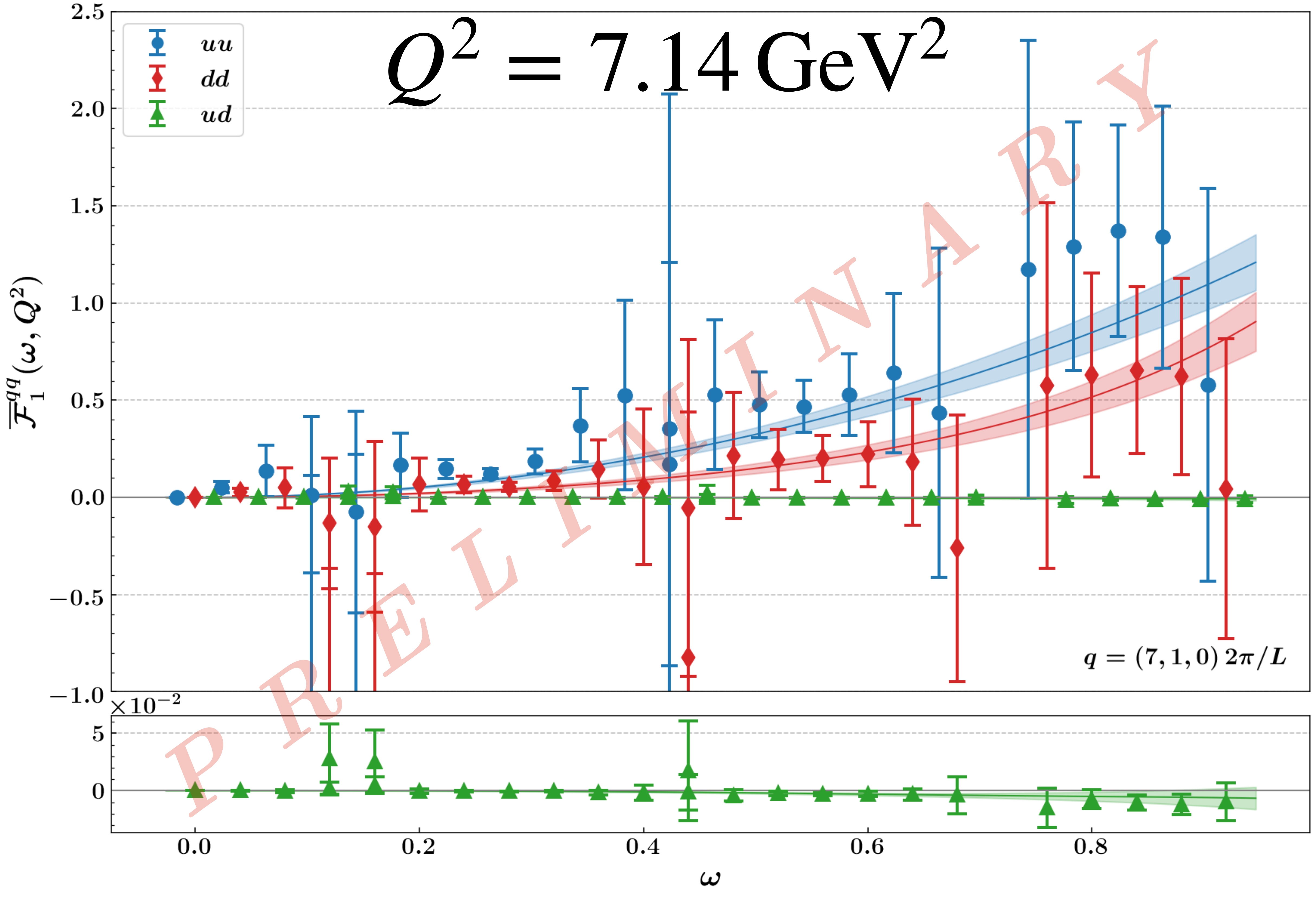
Higher Twist



$a = 0.068$ fm
$m_\pi \sim 410$ MeV
48³x96, 2+1 flavour

- **Twist-4 contributions: *ud* interference term**

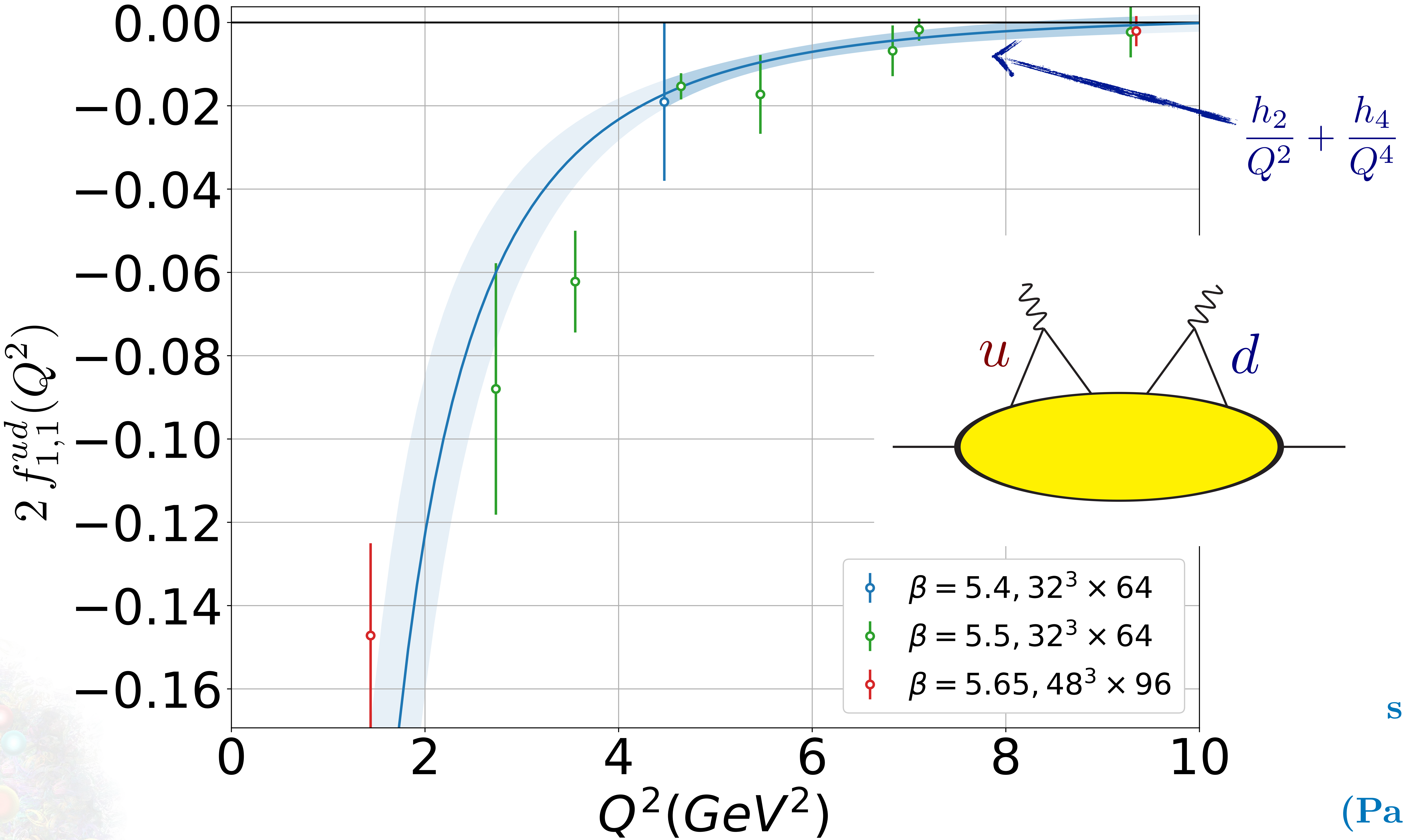




vanishes asymptotically $\sim 1/Q^2$

Higher Twist

Lowest moment of interference T_1



slide courtesy of
Ross Young
(Pacific Spin 2019)

F_2 and F_L

- $\mathcal{F}_2(\omega, Q^2)$

$$T_{\mu\nu}(p,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\mathcal{F}_1(\omega, Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\frac{\mathcal{F}_2(\omega, Q^2)}{p\cdot q}$$

- $\mu = \nu = 3$ and
  $p_z = q_z = 0 \quad \Longrightarrow \quad T_{33}(\omega, Q^2) = -\,g_{33}\mathcal{F}_1(\omega, Q^2)$

- $\mu = \nu = 4$ and $p_4 = iE_N$, $q_4 = 0$:

$$T_{44}(p,q) = -g_{44}\mathcal{F}_1(\omega, Q^2) + \frac{E_N^2}{p\cdot q}\mathcal{F}_2(\omega, Q^2)$$,where $p\cdot q = Q^2\omega/2$

$$\mathcal{F}_2(\omega, Q^2) = \left[T_{44}(p,q) + T_{33}(p,q)\right]\frac{Q^2\omega}{2E_N^2}$$

T_{44} can be extracted via FH approach simply by choosing the temporal components of the currents

F_2 and F_L

- **F_L and the Callan-Gross Relation**

$$F_L(x,Q^2) \equiv \left(1+\frac{4M_N^2x^2}{Q^2}\right)F_2(x,Q^2) - 2xF_1(x,Q^2) \quad \xrightarrow[Q^2\to\infty]{} 0$$

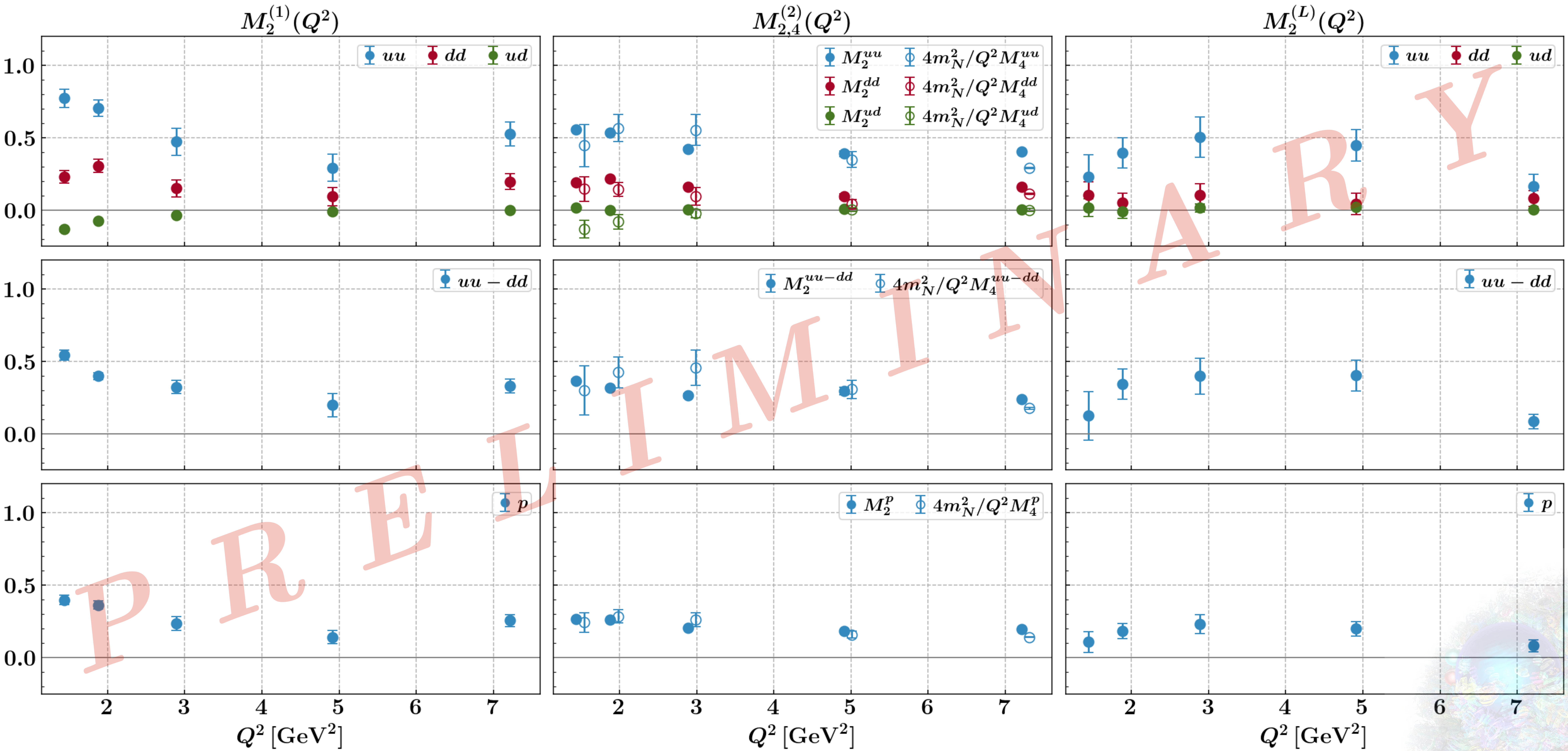
$$\overline{\mathscr{F}}_1(\omega,Q^2) = 4\omega^2\int_0^1 dx\frac{x\,F_1(x,Q^2)}{1-x^2\omega^2} = \sum_{n=1}^{\infty} 2\omega^{2n}M_{2n}^{(1)}(Q^2)\ , \quad M_{2n}^{(1)}(Q^2) = 2\int_0^1 dx\,x^{2n-1}F_1(x,Q^2)$$

$$\mathscr{F}_2(\omega,Q^2) = 4\omega^2\int_0^1 dx\frac{F_2(x,Q^2)}{1-x^2\omega^2} = \sum_{n=1}^{\infty} 4\omega^{2n-1}M_{2n}^{(2)}(Q^2)\, , \quad M_{2n}^{(2)}(Q^2) = \int_0^1 dx\,x^{2n-2}F_2(x,Q^2)$$

$$\boxed{M_2^{(L)}(Q^2) \equiv M_2^{(2)}(Q^2) + \frac{4M_N^2}{Q^2}M_4^{(2)}(Q^2) - M_2^{(1)}(Q^2) \text{ in terms of moments}}$$

Callan-Gross tests

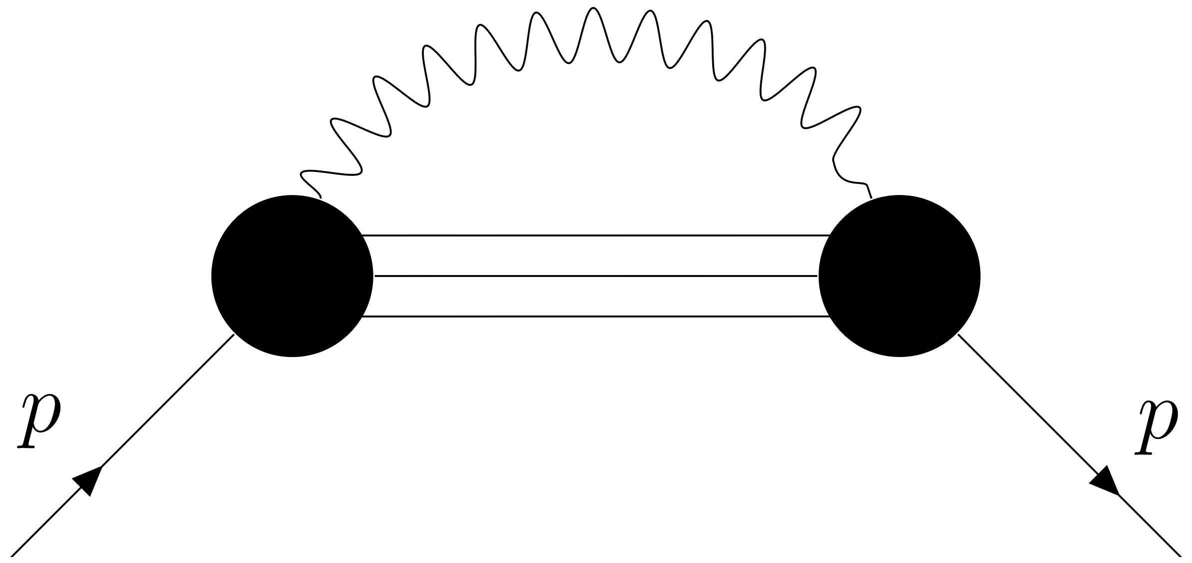
$a = 0.068$ fm
$m_\pi \sim 410$ MeV
48^3x96, 2+1 flavour



Subtraction term

- **Cottingham formula:**

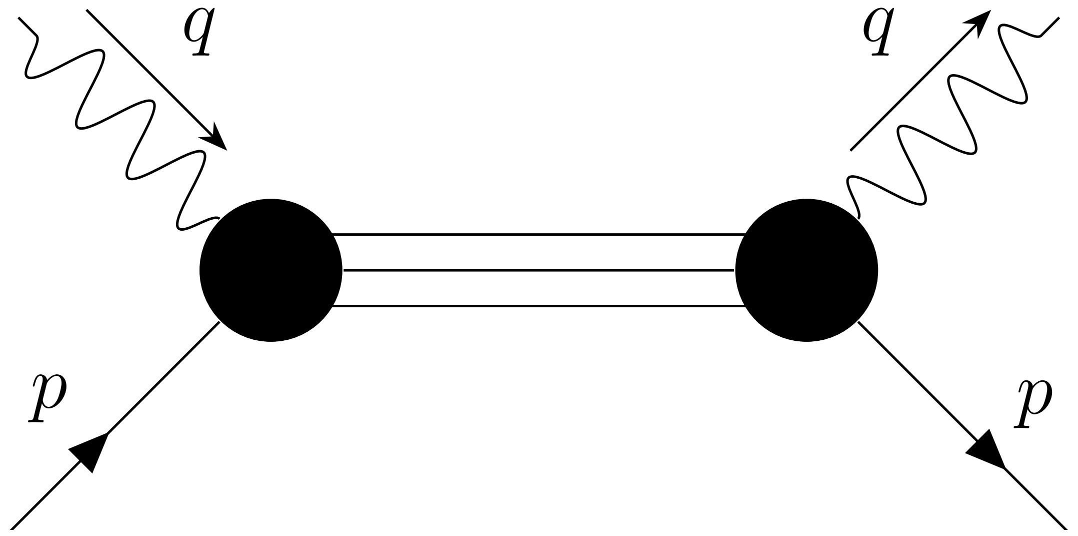
W.N. Cottingham, Annals Phys. 25, 424 (1963)
J. C. Collins, Nucl. Phys., B149:90–100, (1979)
[Erratum: Nucl. Phys.B915,392(2017)]
A. Walker-Loud, C. E. Carlson, G. A. Miller, PRL108, 232301 (2012)



$$\delta M^{\gamma} = \delta M^{\mathrm{el}} + \delta M^{\mathrm{inel}} + \delta M^{\mathrm{sub}} + \delta \tilde{M}^{\mathrm{ct}}$$

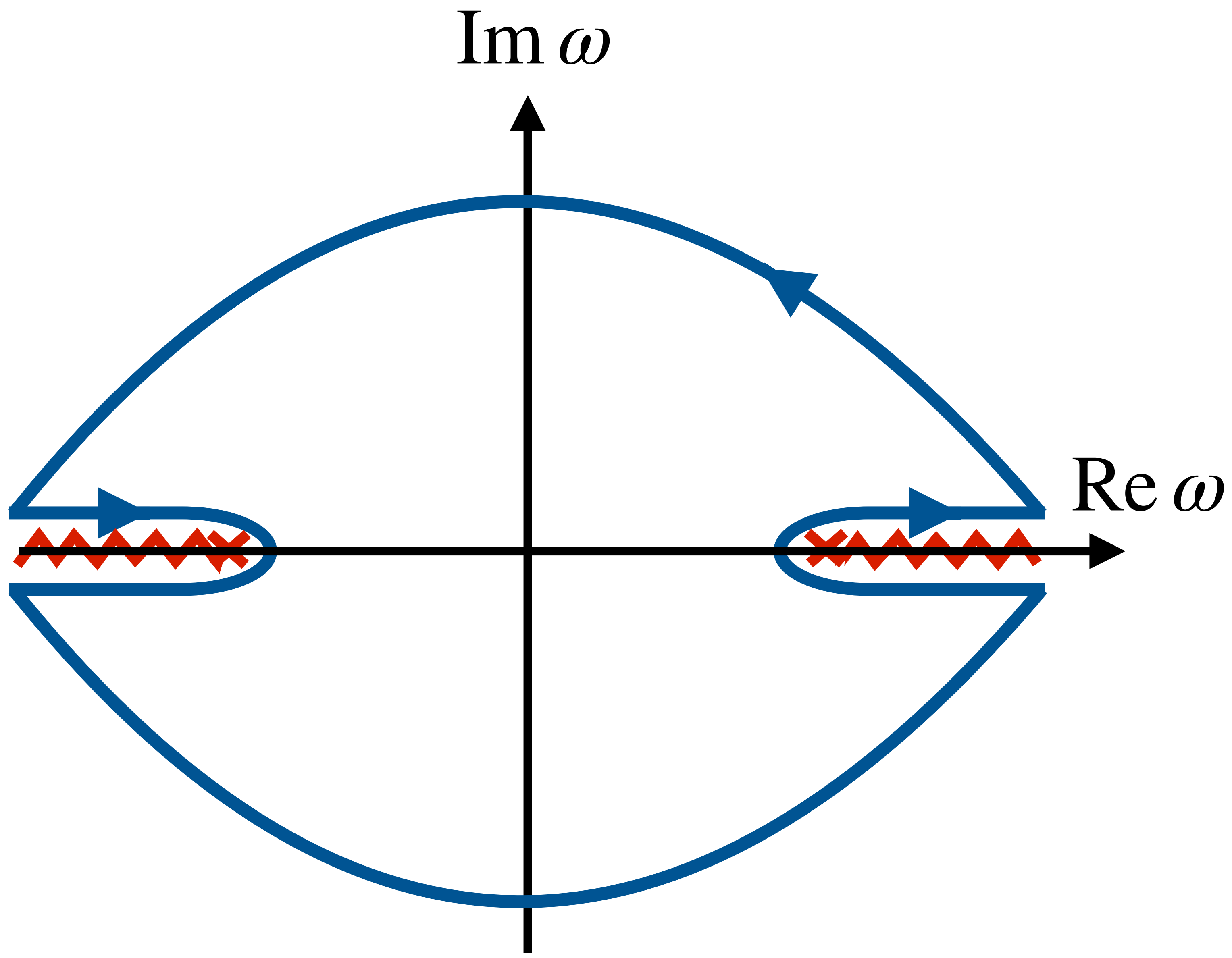
$$\delta M^{\mathrm{sub}} \sim -\frac{3\alpha_{em}}{16\pi M} \int^{\Lambda_0^2} dQ^2 T_1^{p-n}(0, Q^2)$$

EM self energy is related to
the spin-avg. forward Compton amplitude



- **Subtraction term $T_1(0,Q^2)$**

$$\mathscr{F}_1(\omega, Q^2) - \boxed{\mathscr{F}_1(\omega = 0, Q^2)} = \frac{2\omega^2}{\pi} \int_1^{\infty} d\omega' \frac{\mathrm{Im}\, \mathscr{F}_1(\omega', Q^2)}{\omega' \left(\omega'^2 - \omega^2 - i\epsilon\right)}$$



  - dominant uncertainty
    - not accessible via experiments
      - can be calculated via FH approach

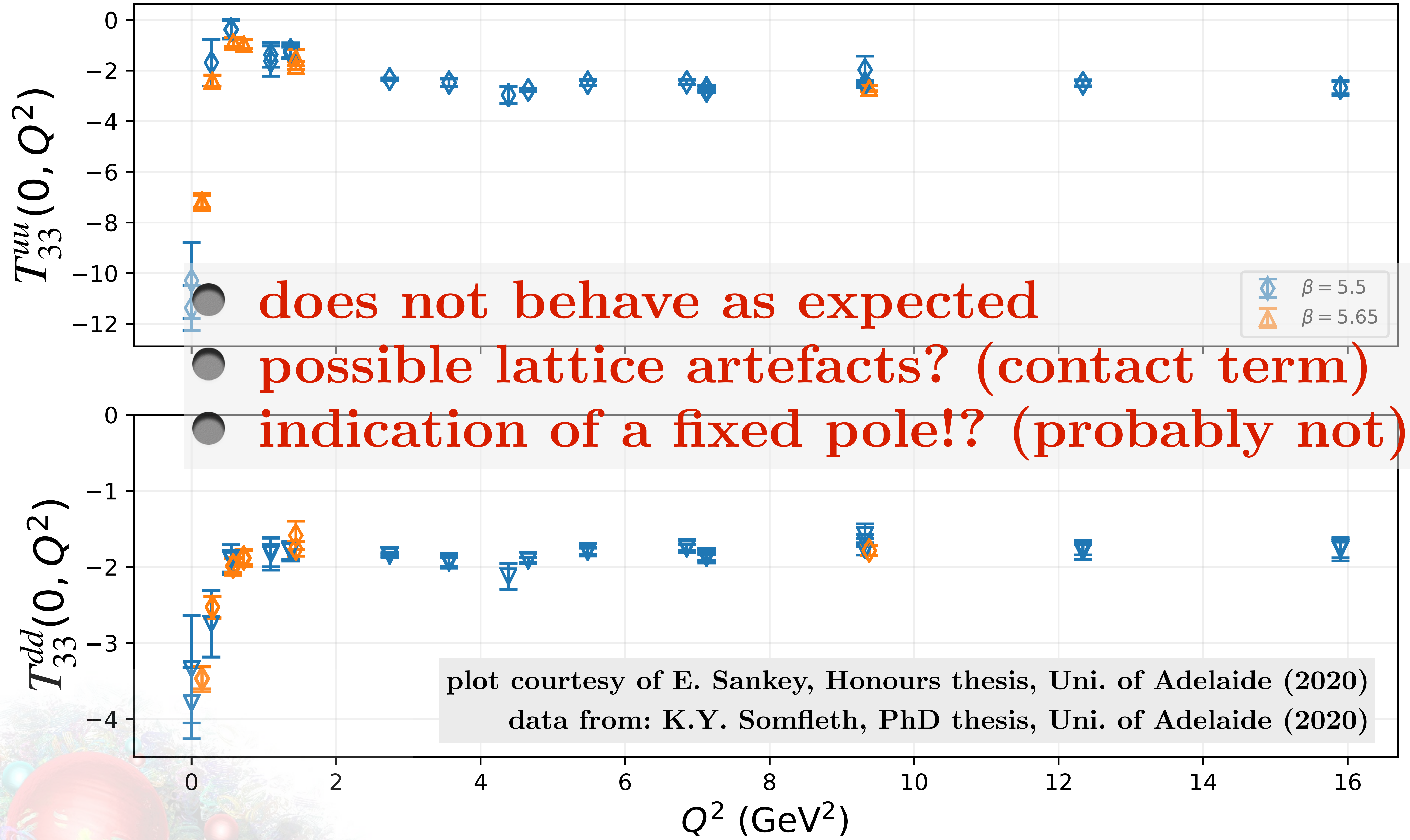
Subtraction term

Recent attention:

- F. Hagelstein and V. Pascalutsa, arXiv:2010.11898 [hep-ph]
- J. Lozano, A. Agadjanov, J. Gegelia, U.-G. Meissner and A. Rusetsky, arXiv:2010.10917 [hep-lat]

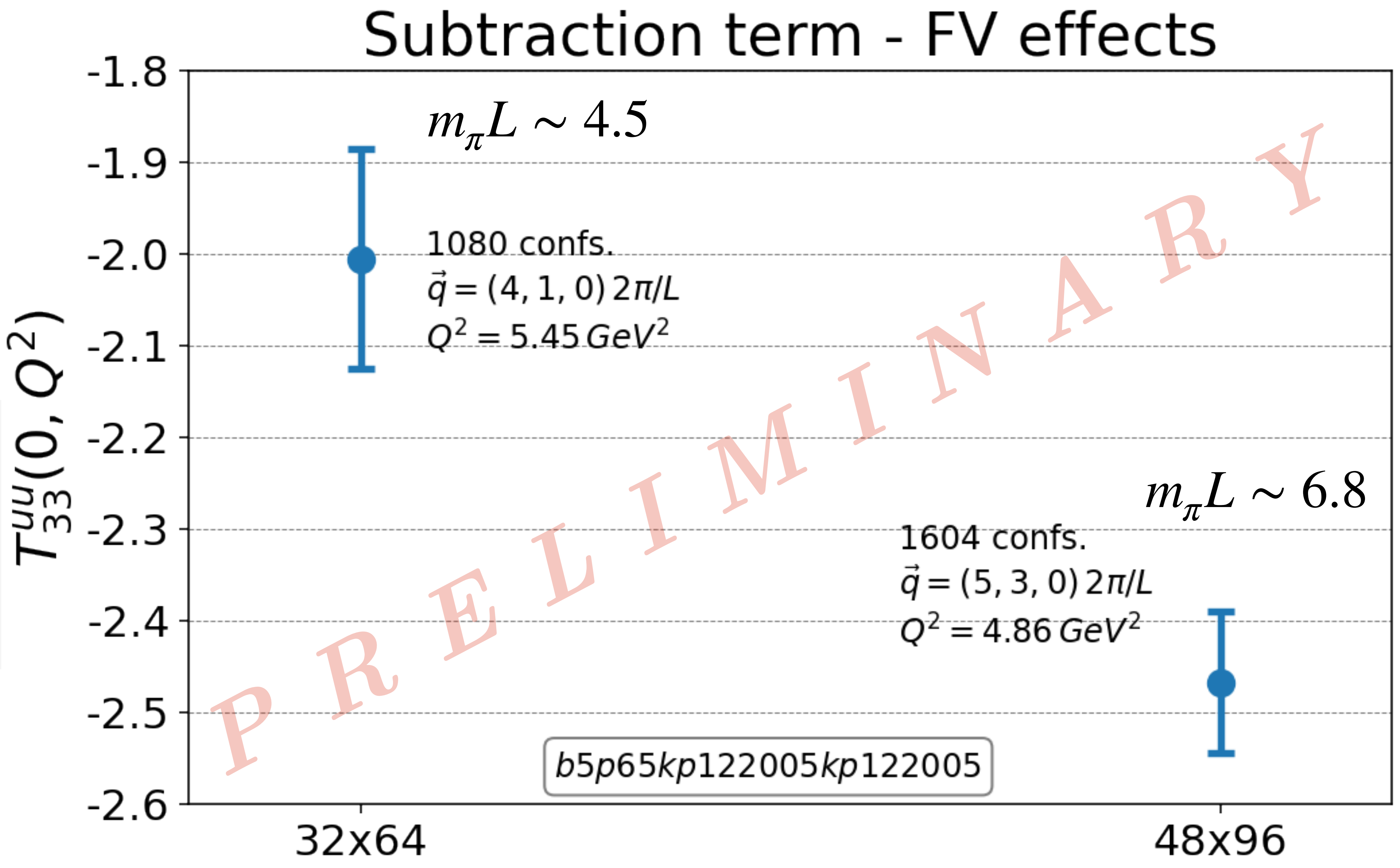
J. C. Collins, Nucl. Phys., B149:90–100, (1979)
[Erratum: Nucl. Phys.B915,392(2017)]

- **Subtraction term ~ $1/Q^2$, OPE expectation**



- does not behave as expected
- possible lattice artefacts? (contact term)
- indication of a fixed pole!? (probably not)

plot courtesy of E. Sankey, Honours thesis, Uni. of Adelaide (2020)
data from: K.Y. Somfleth, PhD thesis, Uni. of Adelaide (2020)



Summary

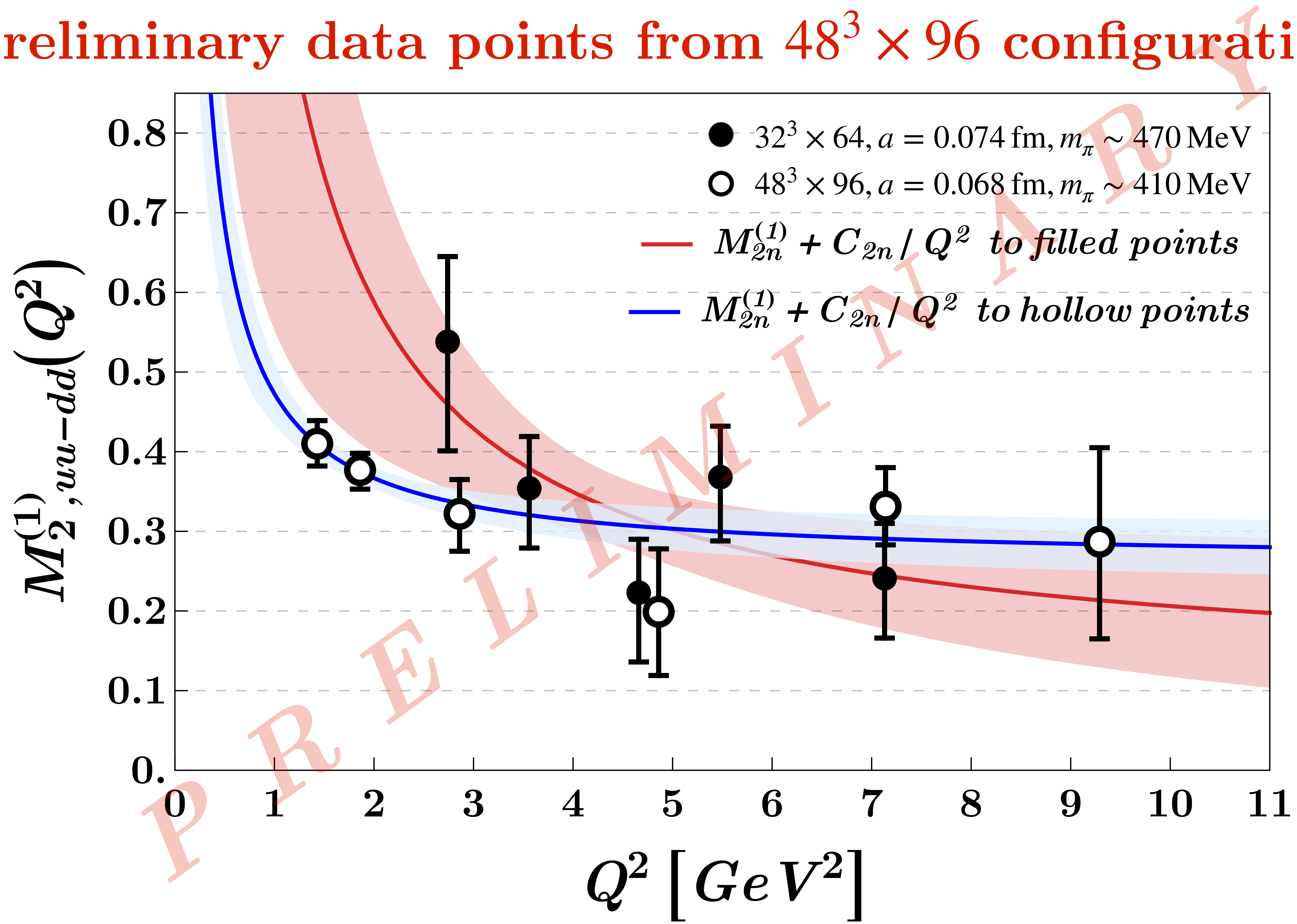
- A new versatile approach!
- Systematic investigation of power corrections, higher-twist effects and scaling is within reach
- Overcomes the operator mixing/renormalisation issues
- Can be extended to:
  - mixed currents, interference terms (work in progress...)
  - spin-dependent structure functions
  - GPDs (A. Hannaford-Gunn, M. Phil. thesis, Uni. of Adelaide (2020))

further questions/comments → kadirutku.can@adelaide.edu.au

Backup Slides

More on Scaling & Power Corr.

- **Preliminary data points from $48^3 \times 96$ configurations**



qualitative comparison
no systematics yet

PDFs

- **determining the PDFs | x-coverage**

$$T_{33}(\omega,Q^2) = \overline{\mathcal{F}}_1(\omega,Q^2) = 4\omega^2\int_0^1 dx\frac{x\,F_1(x,Q^2)}{1-x^2\omega^2}$$ ← formalism in $\boldsymbol{\omega}$ space

$$\equiv \int_0^1 dx K(x,\omega)F_1(x,Q^2),$$ ← back to $\boldsymbol{x}$ space, **inverse problem!**

  - Fredholm integral eq. of the 1st kind: an ill-posed problem

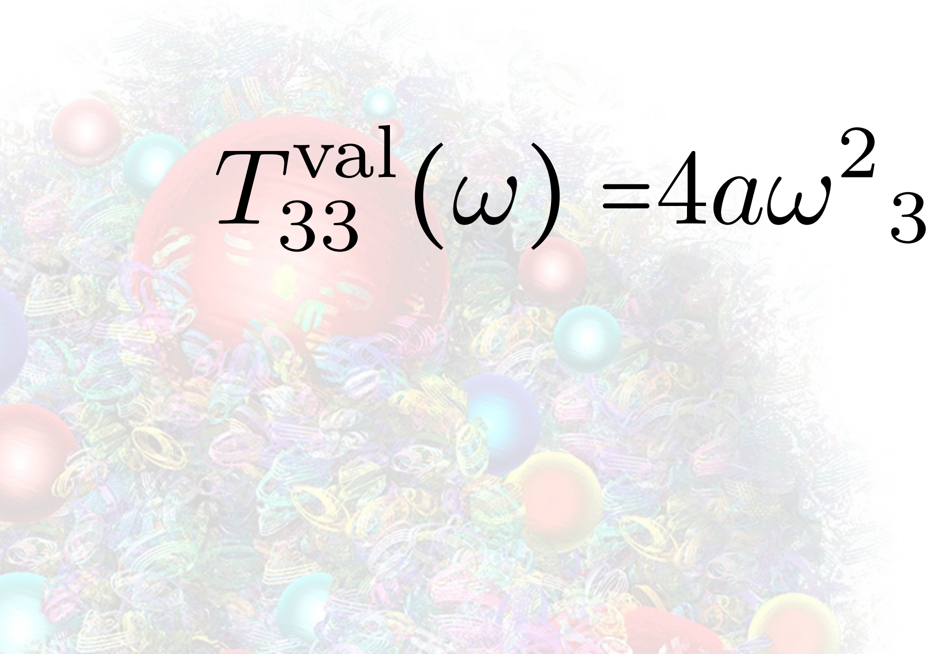
- **starting from the phenom. ansatz**

$$F_1(x,Q^2) \equiv p^{\rm val}(a,b,c) = \frac{a\,x^b\,(1-x)^c\Gamma(b+c+3)}{\Gamma(b+2)\Gamma(c+1)}$$

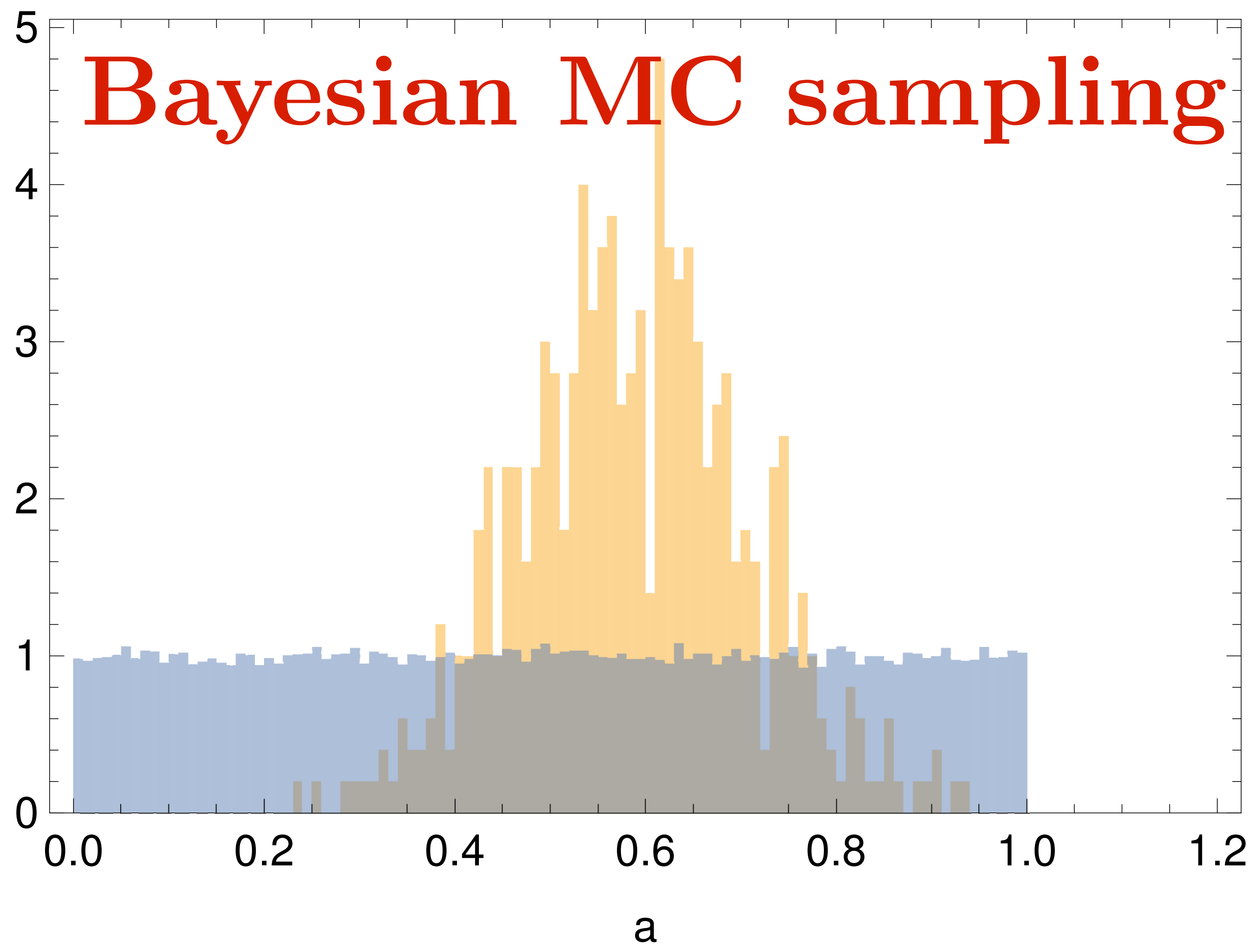
evaluate the dispersion integral

$$T_{33}^{\rm val}(\omega) = 4a\omega^2\,{}_3F_2\left[\begin{matrix}1,\ (b+2)/2,\ (b+3)/2\\ (b+c+3)/2,\ (b+c+4)/2\end{matrix};\omega^2\right] = 4\,a\,\omega^2\,(\ c_0(a,b,c)+c_1(a,b,c)\,\omega^2 + c_2(a,b,c)\,\omega^4+\cdots+c_n(a,b,c)\,\omega^{2n}+\cdots)$$

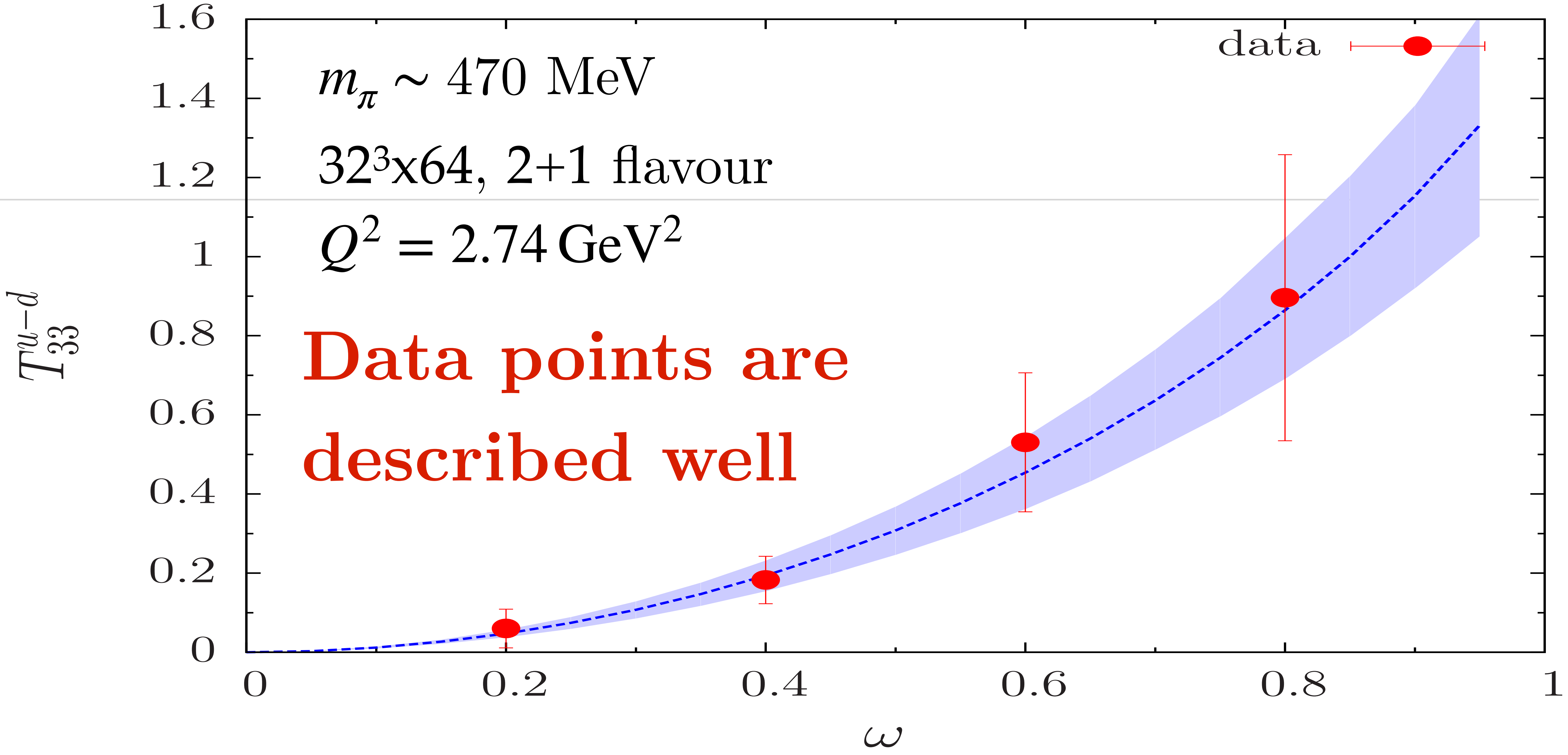
generalised hypergeometric function



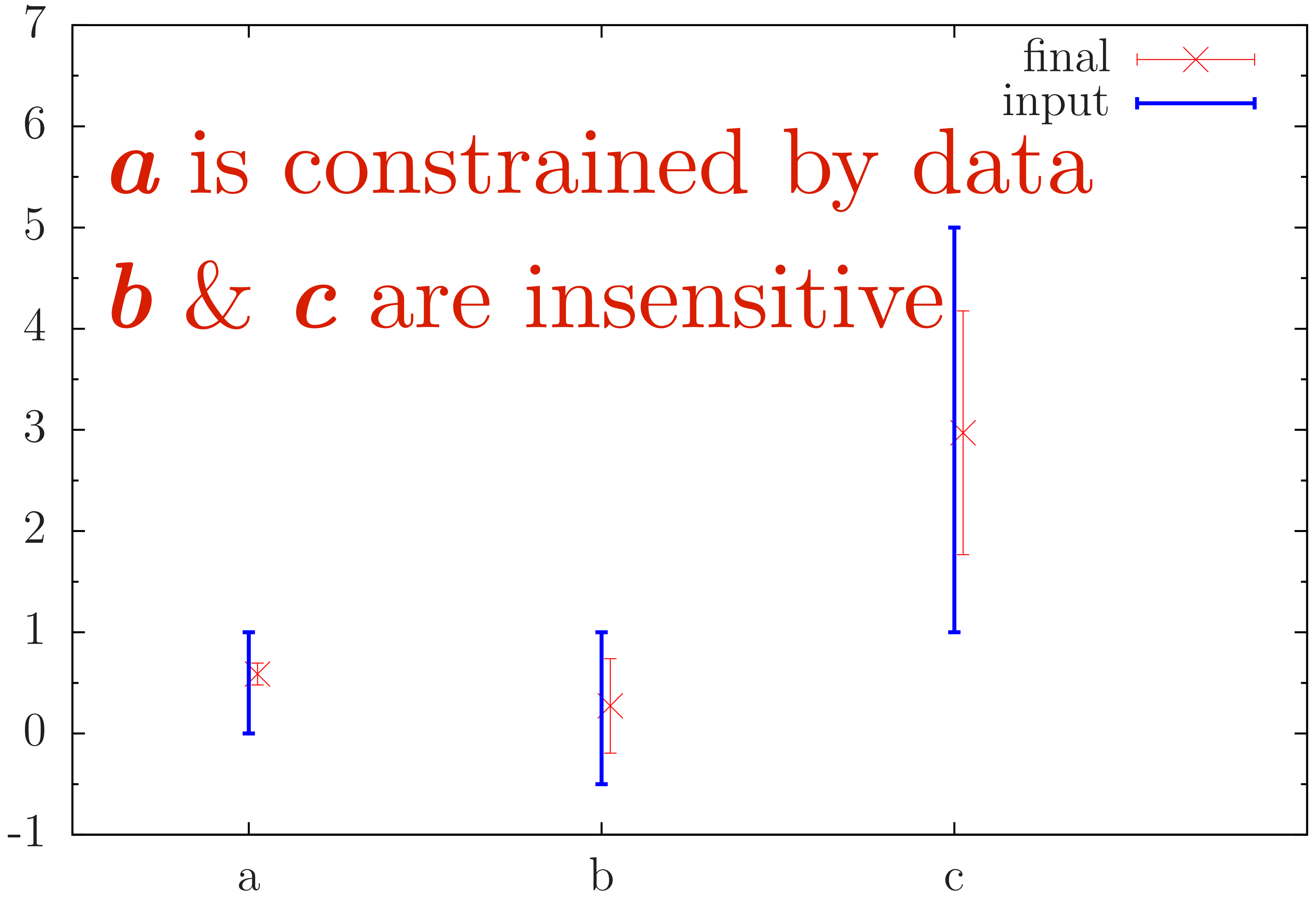
PDFs



Bayesian MC sampling

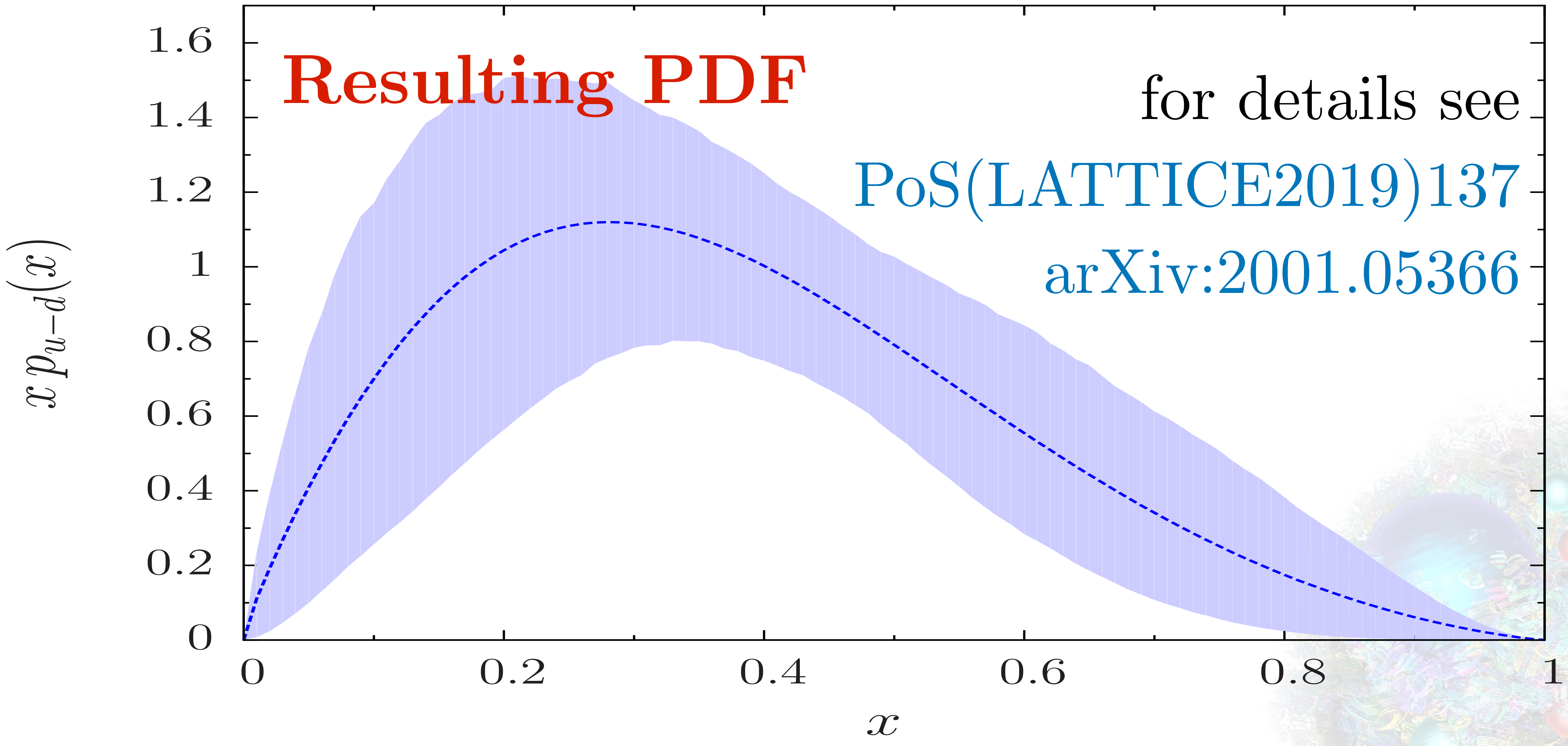


Data points are described well



a **is constrained by data**

b **&** ***c*** **are insensitive**



Resulting PDF

for details see

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