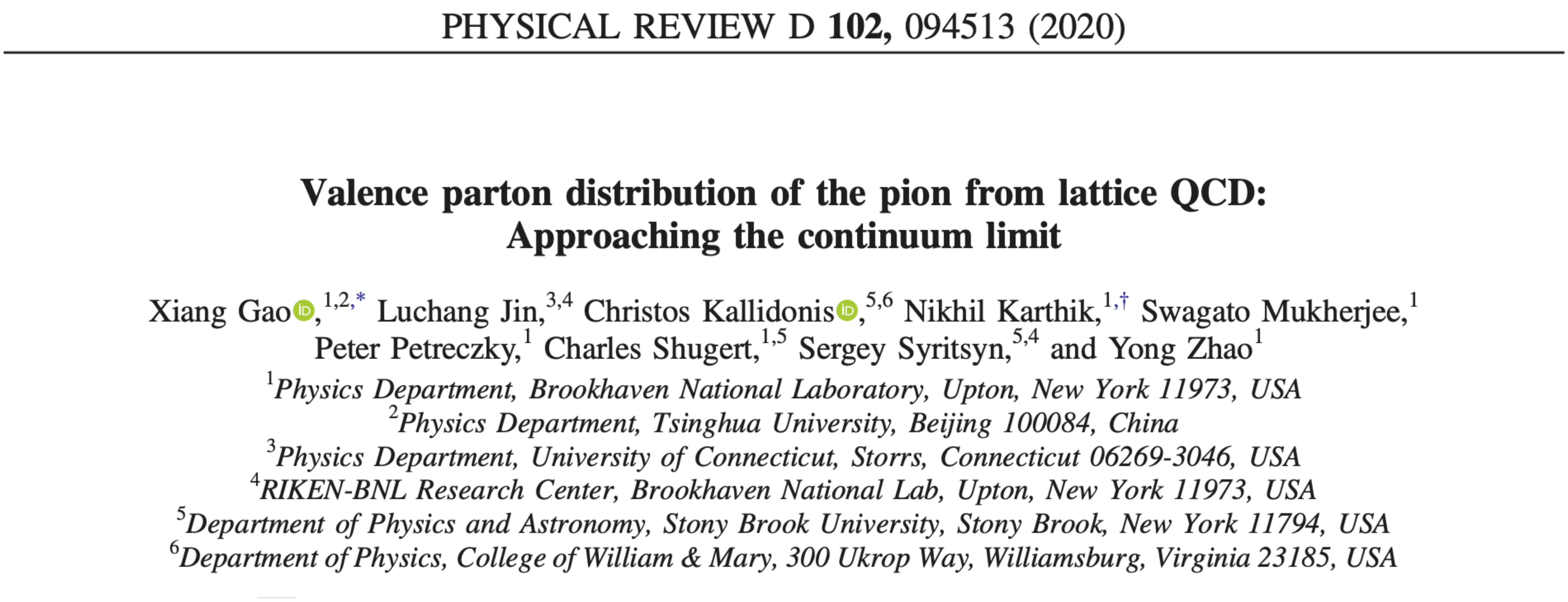

PHYSICAL REVIEW D **102,** 094513 (2020)

Valence parton distribution of the pion from lattice QCD: Approaching the continuum limit

Xiang Gao,[1,2,*] Luchang Jin,[3,4] Christos Kallidonis,[5,6] Nikhil Karthik,[1,†] Swagato Mukherjee,[1]
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ANL journal club

June 8, 2021

Nikhil Karthik

W&M - JLab

Outline of the paper

Valence PDF of $\pi^+(u\overline{d})$

$$f_v^\pi(x) = \underbrace{f_u(x)}_{\text{(sea+valence)}} - \underbrace{f_{\bar{u}}(x)}_{\text{(sea)}} = \underbrace{f_u(x) - f_d(x)}_{\text{(Isospin symmetry)}}, \quad 0 < x < 1$$

Matrix element that is used in the paper: bilocal operator that enters quasi-PDF, pseudo-PDF

$$\langle \pi, P_z \big| \overline{\psi}(z) W_z(z,0) \tau_3 \gamma_t \psi(0) \big| \pi, P_z \rangle = 2E(P_z)\mathcal{M}(zP_z, z^2; p^R)$$

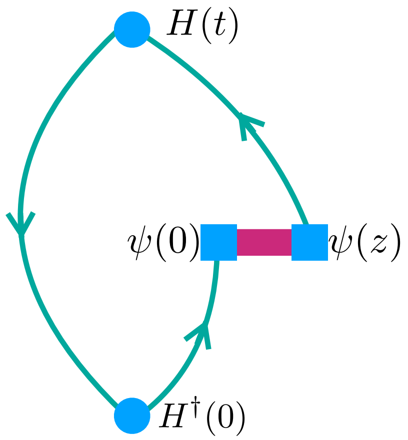
Non-singlet:
a)No gluon mixing;
b) No disconnected pieces

No mixing
with scalar

X. Ji, Phys. Rev. Lett. **110**, 262002 (2013), arXiv:1305.1539 [hep-ph].

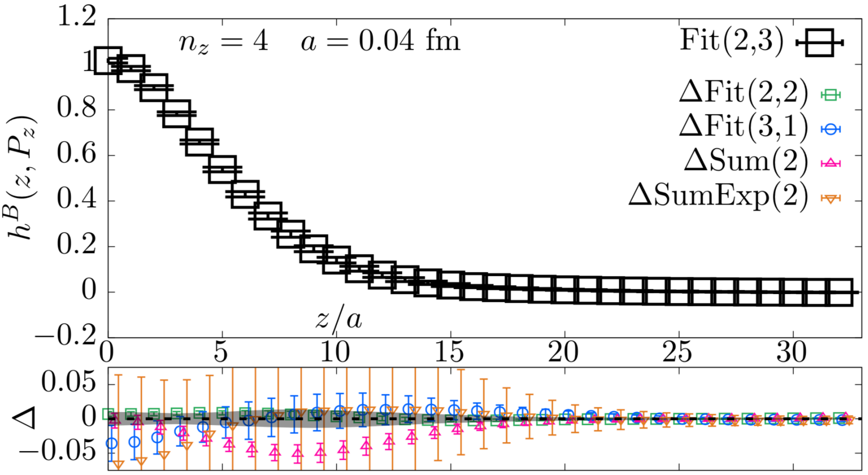
A. Radyushkin, Phys. Rev. D **96**, 034025 (2017), arXiv:1705.01488 [hep-ph].

Algorithm



$$\frac{\left\langle H(t_{\text{sink}})\ \overset{\overline{\psi}(0)}{\bullet}\underset{W(0,z)}{\rule{3em}{0.2em}}\overset{\gamma_t\psi(z)}{\bullet}\ H^\dagger(0)\right\rangle}{\left\langle H(t_{\text{sink}})H^\dagger(0)\right\rangle}$$

$t_{\text{sink}} \to \infty$



$$h(z,P_z) = \left\langle P_z \middle|\ \overset{\overline{\psi}(0)}{\bullet}\underset{W(0,z)}{\rule{3em}{0.2em}}\overset{\gamma_t\psi(z)}{\bullet}\ \middle| P_z \right\rangle$$

Renormalize

$$\mathcal{M}(zP_z, z^2) = \frac{h(z,P_z)}{h(z,0)}$$

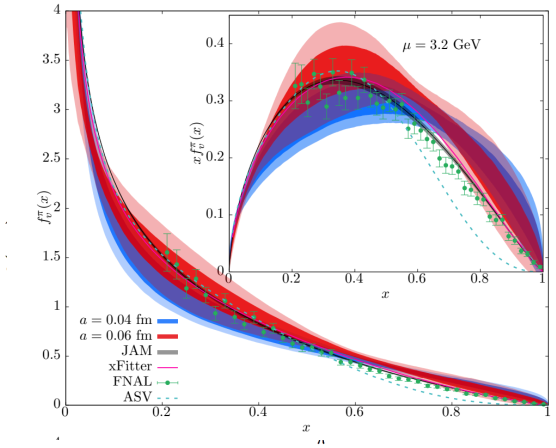
$\langle x^n \rangle$ ← Access moments directly

Twist-2 OPE (LaMET)

$$\sum_n \boxed{\langle x^n\rangle} c_n(\mu^2 z^2)\frac{(-iP_z z)^n}{n!}$$

Fit ansatz

$$f(x,\mu^2)$$



Lattice details

Ensemble details and statistics

Ensemble: 2+1 flavor HISQ ⟶ A. Bazavov *et al.* (HotQCD), Phys. Rev. **D90**, 094503 (2014), arXiv:1407.6387 [hep-lat].

Sea quark mass: $m_\pi = 160\ \mathrm{MeV}$

Valence: tadpole-improved Wilson-Clover fermions coupled to 1-HYP links

Valence quark mass: $m_\pi = 300\ \mathrm{MeV}$

ensemble a, $L_t \times L^3$	$m_q a$	$m_\pi L_t$	n_z	z range	#cfgs	(#ex,#sl)
$a =$ 0.06 fm, 64×48^3	-0.0388	5.85	0,1	[0,15]	100	$(1, 32)$
			2,3,4,5	[0,8]	525	$(1, 32)$
				[9,15]	416	$(1, 32)$
				[16,24]	364	$(1, 32)$
$a =$ 0.04 fm, 64×64^3	-0.033	3.90	0,1	[0,32]	314	$(3, 96)$
			2,3	[0,32]	314	$(4, 128)$
			4,5	[0,32]	564	$(4, 128)$

← AMA

Statistics increase

Momenta and quark smearing

n_z	P_z (GeV)		ζ
	$a = 0.06$ fm	$a = 0.04$ fm	$= k_z/P_z$
0	0	0	0
1	0.43	0.48	0
2	0.86	0.97	1
3	1.29	1.45	2/3
4	1.72	1.93	3/4
5	2.15	2.42	3/5

Momentum smearing:

G. S. Bali, B. Lang, B. U. Musch, and A. Schäfer, Phys. Rev. D **93**, 094515 (2016), arXiv:1602.05525 [hep-lat].

Details in prequel:

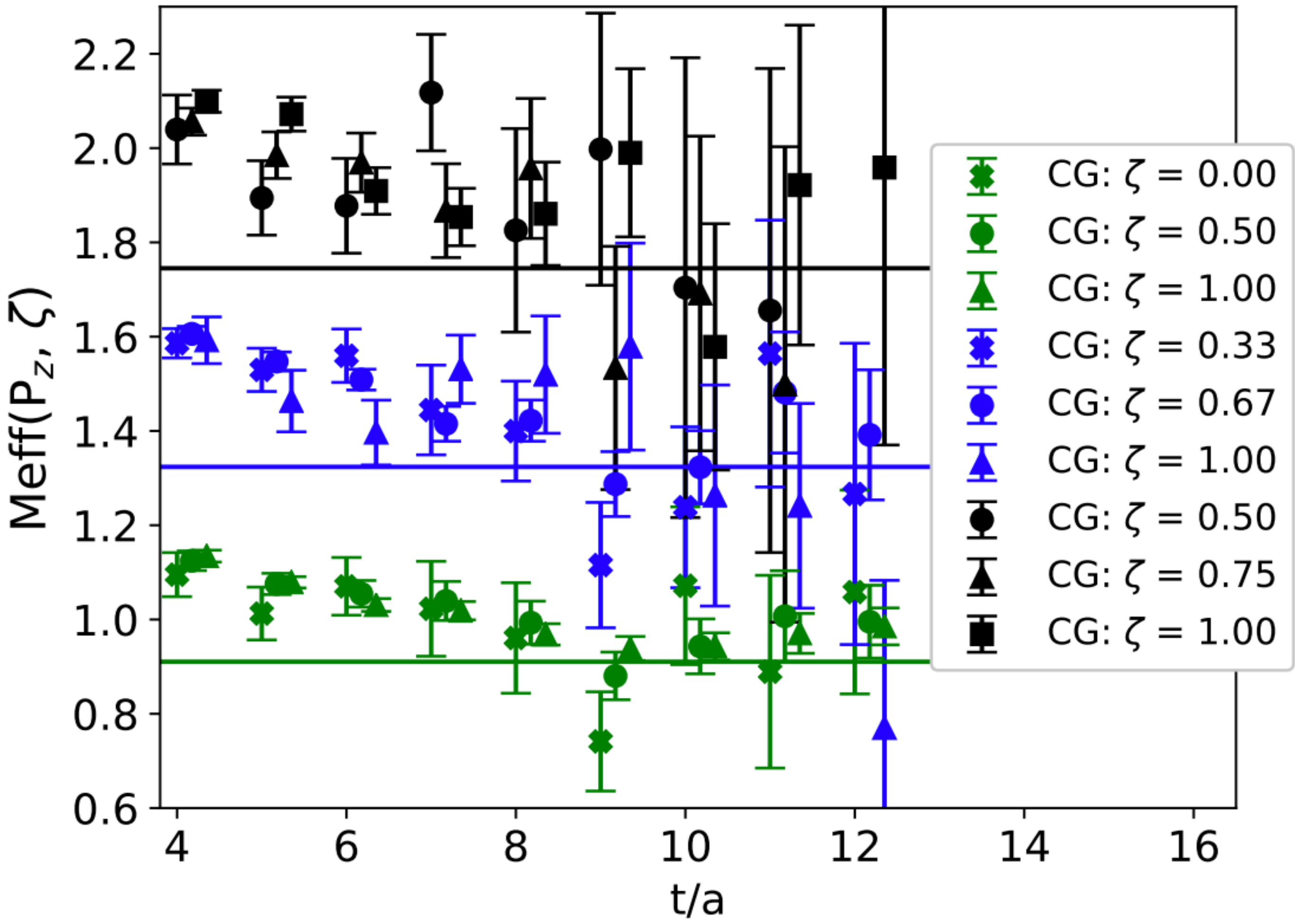
T. Izubuchi, L. Jin, C. Kallidonis, N. Karthik, S. Mukherjee, P. Petreczky, C. Shugert, and S. Syritsyn, Phys. Rev. D **100**, 034516 (2019), arXiv:1905.06349 [hep-lat].

Coulomb gauge-fixed Gaussian smearing

$$\psi_{\rm smear} = (\mathcal{S}^{(\vec{k})}\psi)_x$$

$$\mathcal{S}^{(\vec{k})}_{\vec{x},\vec{y}} = \sum_{\vec{p}} e^{i\vec{p}(\vec{x}-\vec{y})}\, e^{-\frac{1}{2}w^2_{CG}(\vec{p}-\vec{k})^2}$$

(tuning results for Momentum smearing)

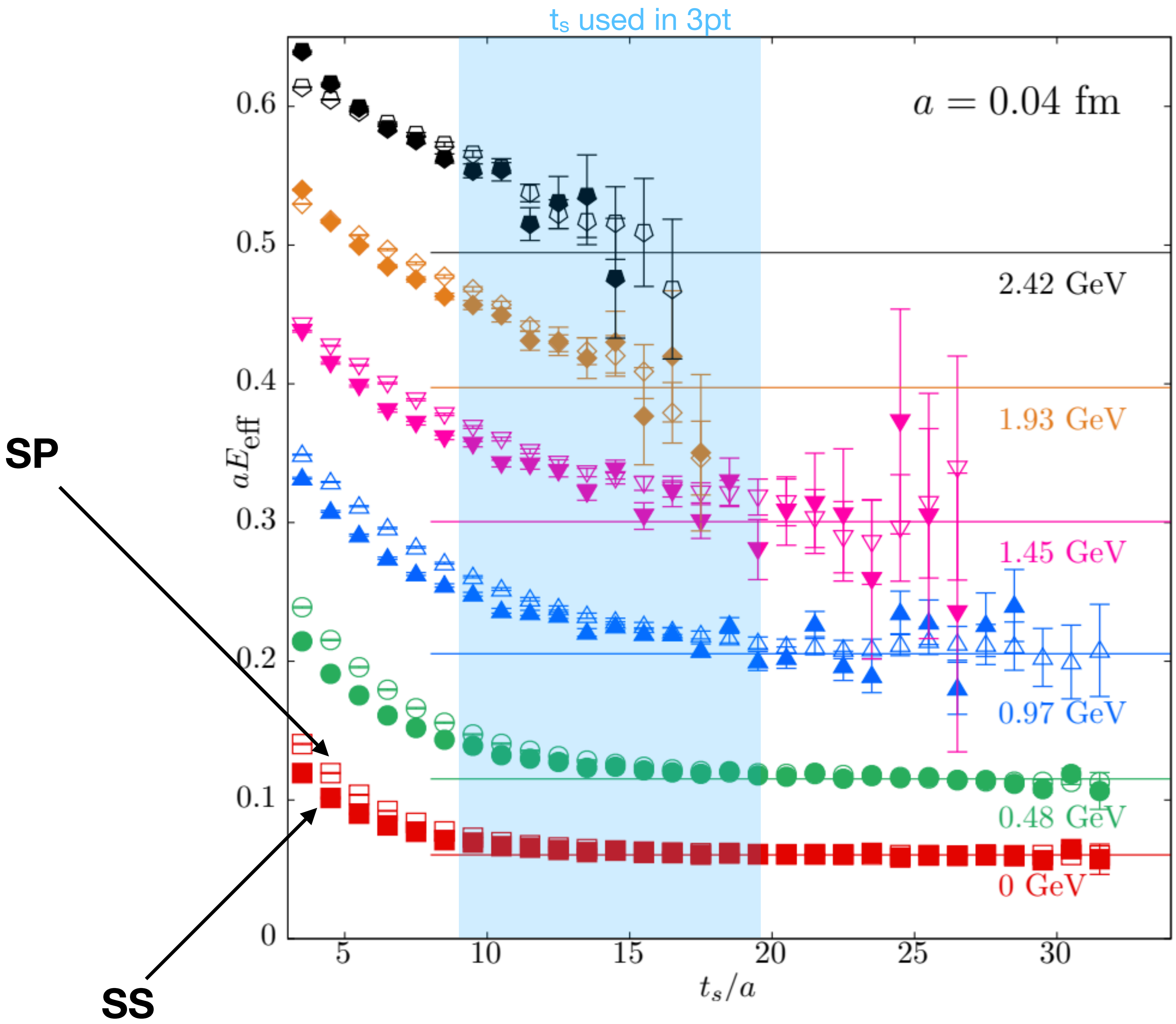


2-pt function analysis:

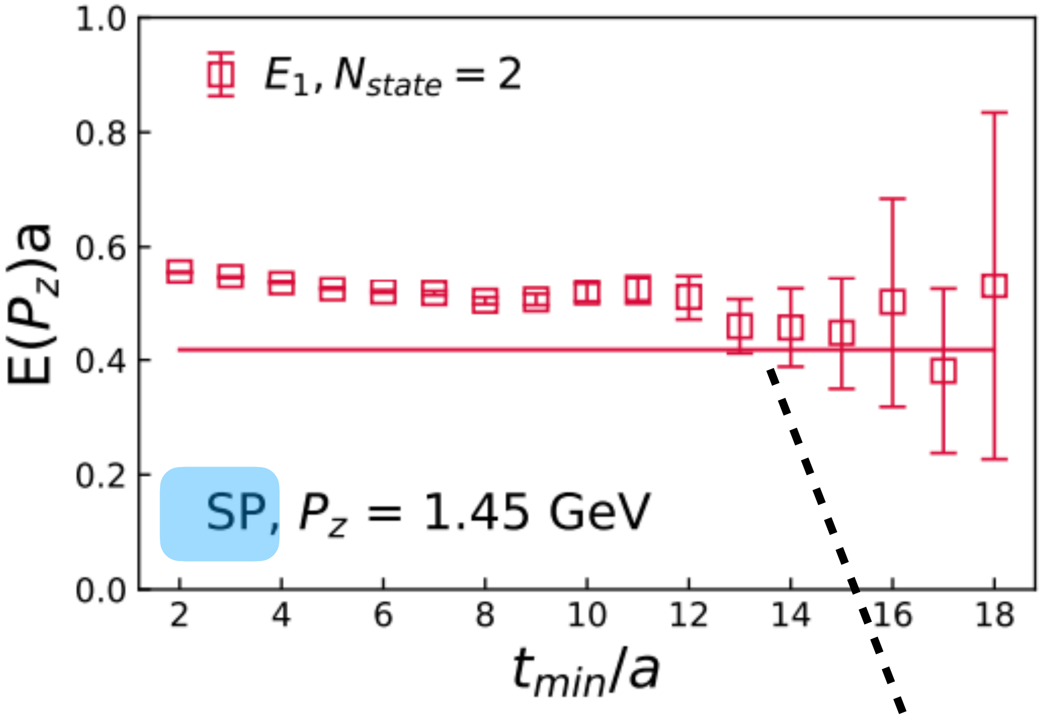
Determine the spectral data to fit the 3pt function

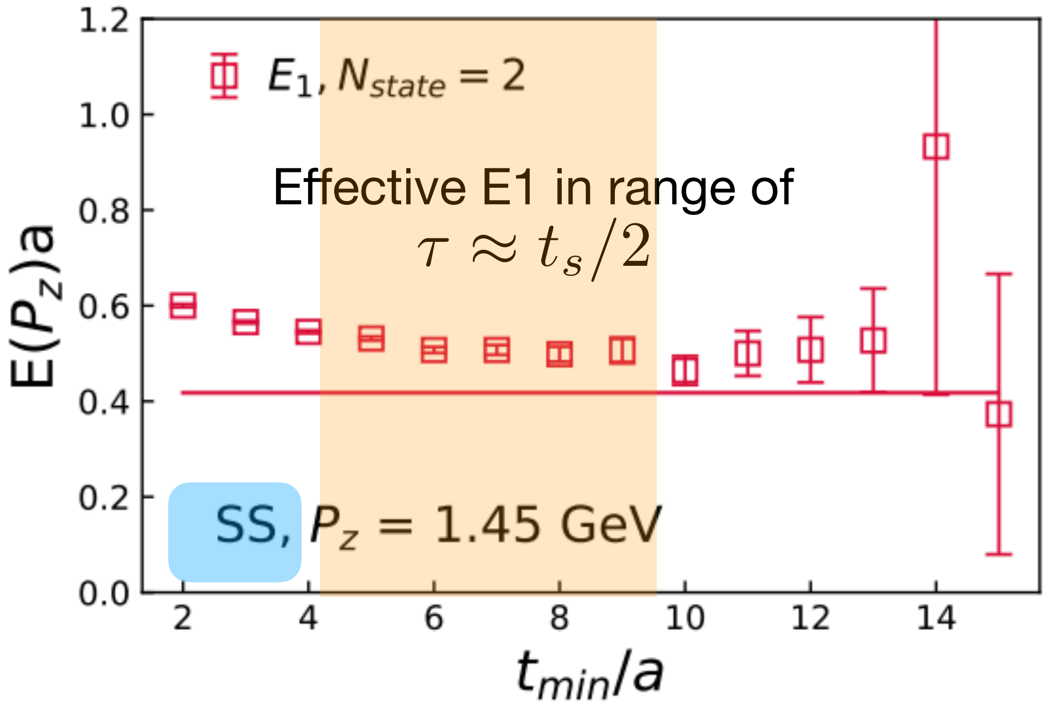
$$C_{2\mathrm{pt}}^{ss'}(t_s;P_z) = \langle \pi_{s'}(\mathbf{P},t_s)\pi_s^\dagger(\mathbf{P},0)\rangle \,.$$

Used *smeared-point* (SP) and *smeared-smeared* (SS) correlators



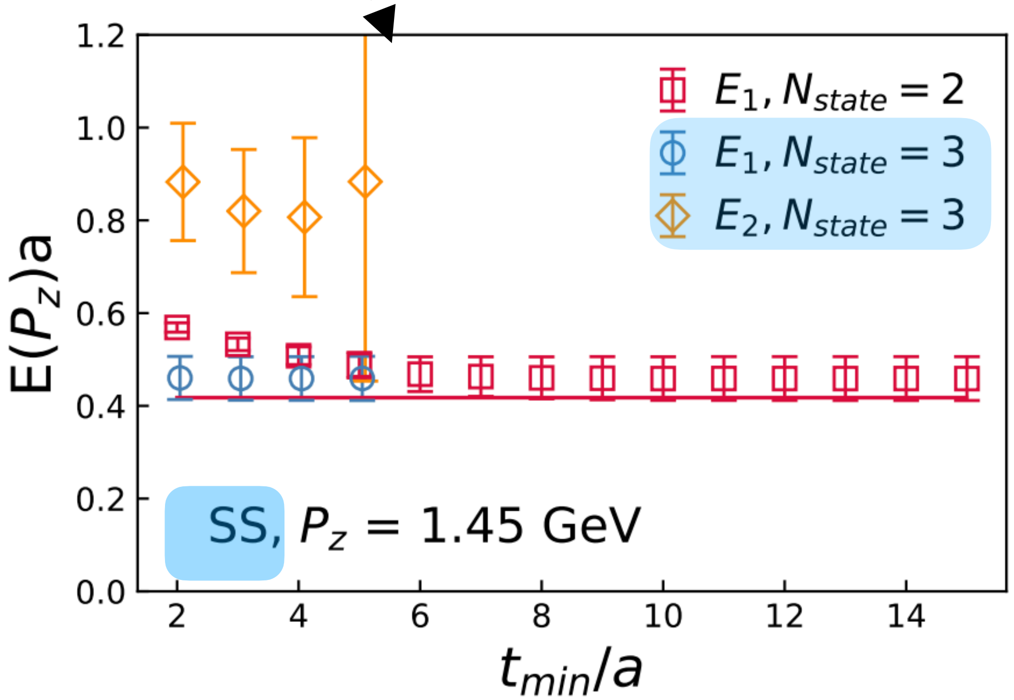
Making use of SP and SS to determine excited states via 2-state and 3-state fits



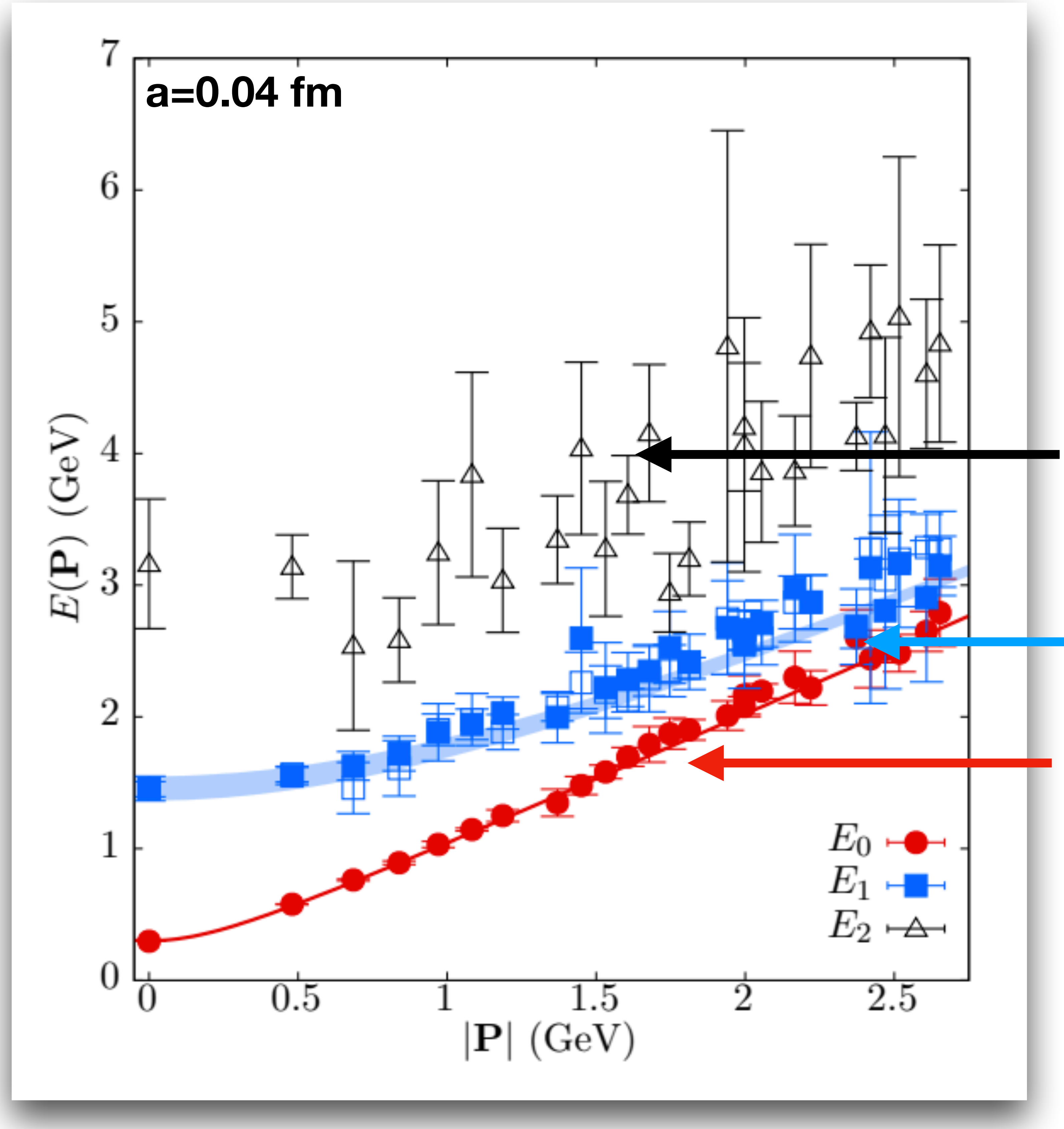


Effective E1 in range of $\tau \approx t_s/2$

Prior for E_1 from 2-state SP to 3-state SS
Use dispersion relation for E_0



Two-point function analysis



Some effective 2nd state

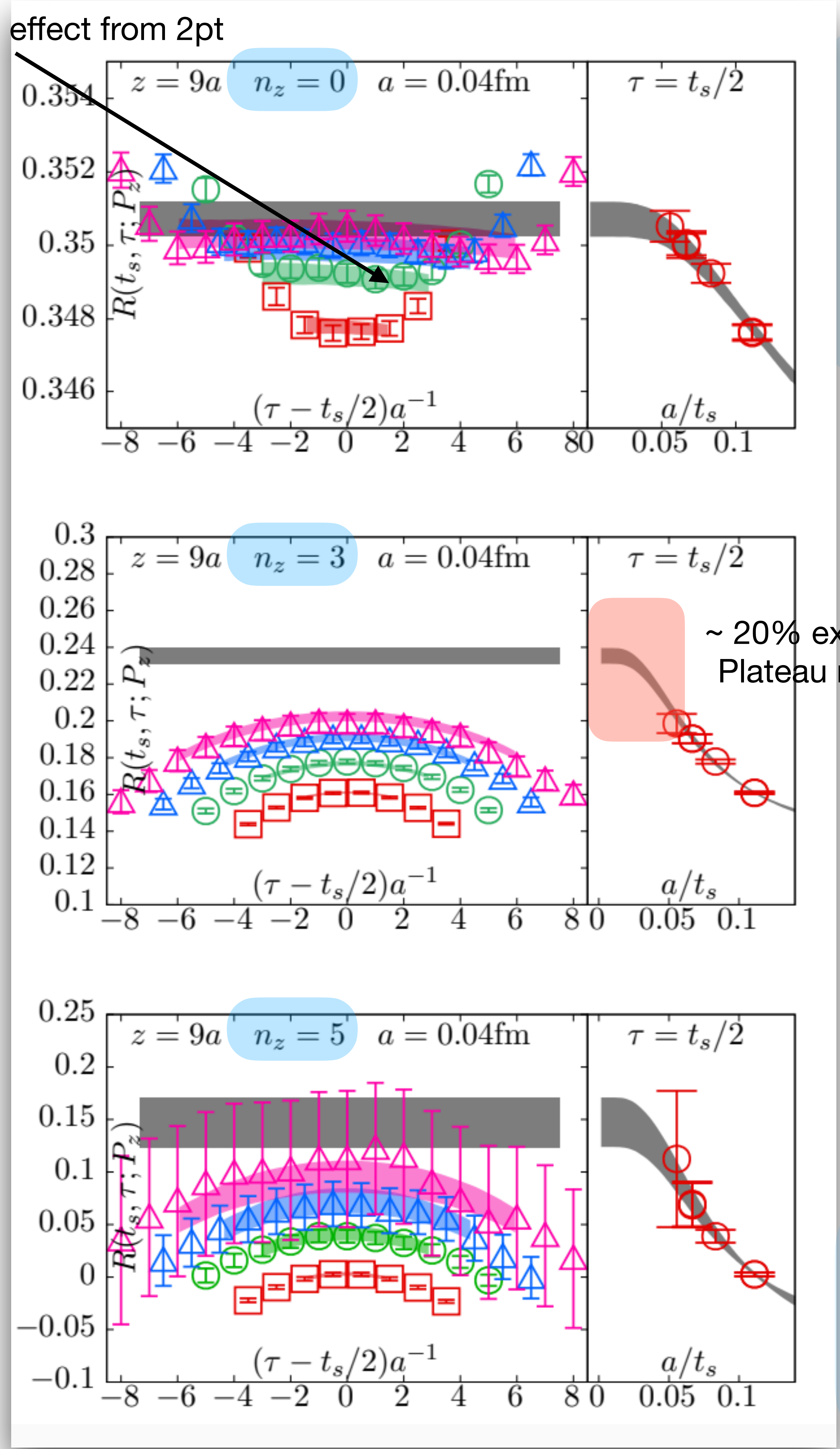
Most-Likely pi(1300)

Pion

*more momenta added later in PRD03 (2021) 9, 094510

3-pt function analysis:

Extrapolate to get bare matrix element

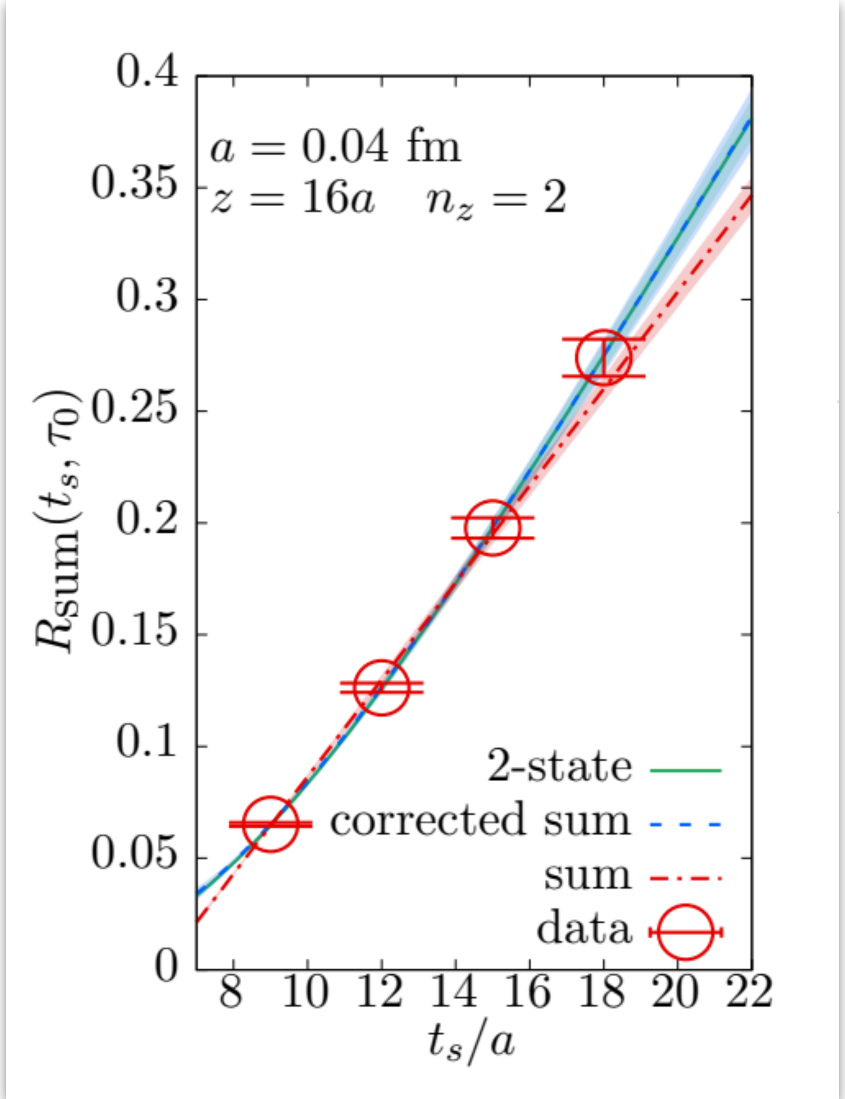


Periodicity effect from 2pt

~ 20% extrapolations
Plateau not feasible

$$R(t_s,\tau;z,P_z) \equiv \frac{C_{3\mathrm{pt}}(t_s,\tau;z,P_z)}{C_{2\mathrm{pt}}(t_s;P_z)}.$$

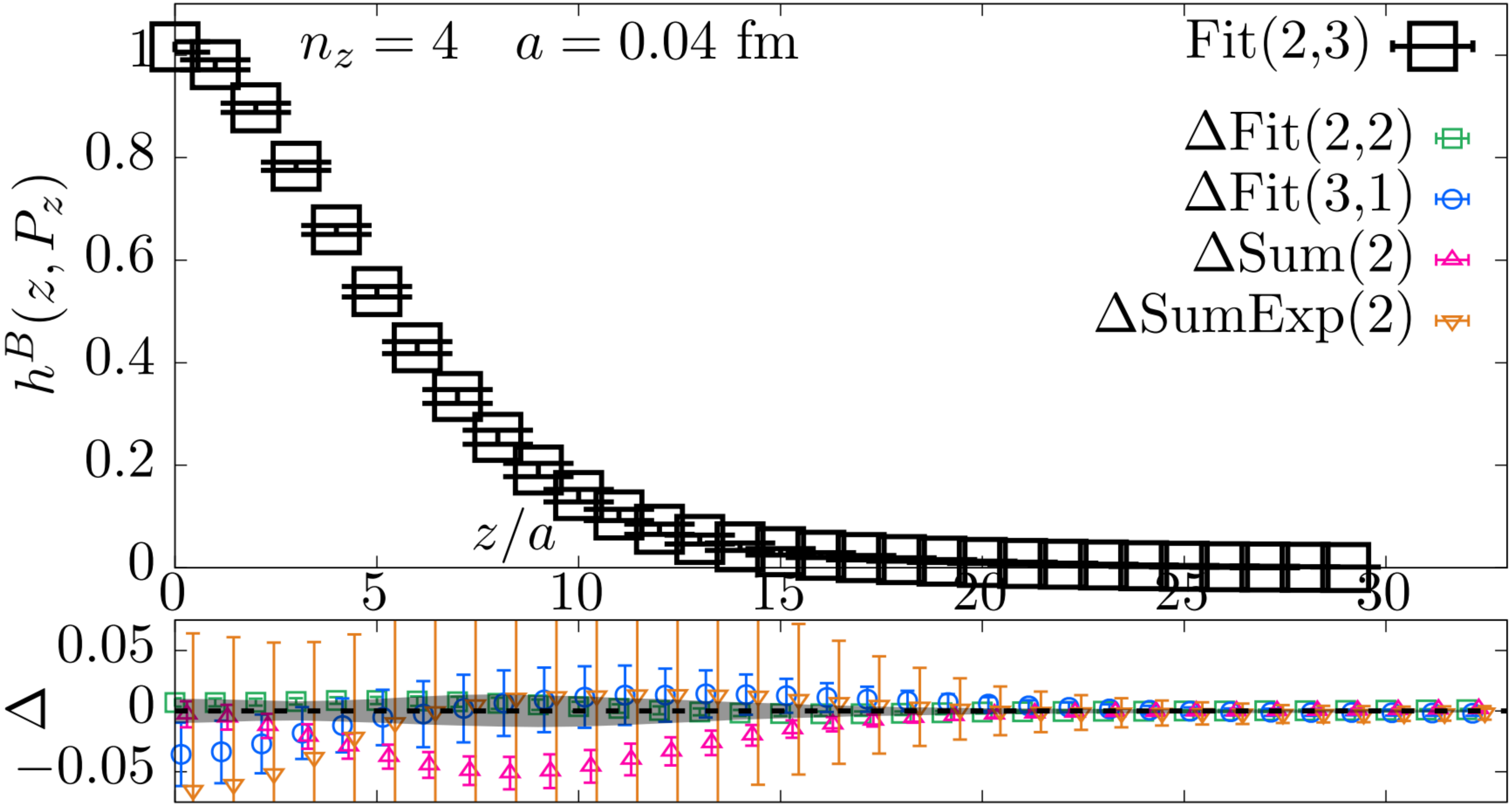
$$R(t_s,\tau) = \frac{\sum_{n,n'}^{N} A_n A_{n'}^{*} \langle E_n,P|O_{\gamma_t}(z)|E_{n'},P\rangle \, e^{-(E_{n'}-E_n)\tau - E_n t_s}}{\sum_{m}^{N} |A_m|^2 e^{-E_m t_s}}$$



$$R_{\mathrm{sum}}(t_s) = \sum_{\tau=n_{\mathrm{sk}}a}^{t_s-n_{\mathrm{sk}}a} R(t_s,\tau).$$

$$R_{\mathrm{sum}}(t_s) = (t_s - 2n_{\mathrm{sk}}a)h^B(z,P_z) + B_0 + \cancel{B_1 e^{-(E_1-E_0)t_s}}. \quad (13)$$

Looking for robustness in extrapolations



Difference in extrapolated values from Fit(2,3)

Renormalization and matching

Renormalization methods used:

***Double ratio:**
make z=0 matrix element to be 1 by hand

RI-MOM

(p-slash, Landau gauge)

J.-W. Chen, T. Ishikawa, L. Jin, H.-W. Lin, Y.-B. Yang, J.-H. Zhang, and Y. Zhao, Phys. Rev. D **97**, 014505 (2018), arXiv:1706.01295 [hep-lat].

$$h_0^R(z, P_z, P^R) = Z_q Z_{\gamma_t\gamma_t}(z, P^R) h^B(z, P_z) \xrightarrow[\text{Ratio}]{\text{Double}} h^R(z, P_z, P^R) \equiv \frac{h_0^R(z, P_z, P^R)}{h_0^R(0, P_z, P^R)}.$$

RGI Ratio and its generalization

K. Orginos, A. Radyushkin, J. Karpie, and S. Zafeiropoulos, Phys. Rev. D **96**, 094503 (2017), arXiv:1706.05373 [hep-ph].

$$\mathcal{M}_0(z, P_z, P_z^0) = \frac{h^B(z, P_z)}{h^B(z, P_z^0)}. \xrightarrow[\text{Ratio}]{\text{Double}} \mathcal{M}(z, P_z, P_z^0) = \frac{\mathcal{M}_0(z, P_z, P_z^0)}{\mathcal{M}_0(0, P_z, P_z^0)}$$

Why use non-zero P_z^0 ?

- Avoid wrap-around effect
- Make leading-twist dominate both numerator and denominator

Leading-twist OPE expression for ratio

Ratio and its generalization

$$\mathcal{M}_0(z, P_z, P_z^0) = \frac{h^B(z, P_z)}{h^B(z, P_z^0)}. \xrightarrow[\text{Ratio}]{\text{Double}} \mathcal{M}(z, P_z, P_z^0) = \frac{\mathcal{M}_0(z, P_z, P_z^0)}{\mathcal{M}_0(0, P_z, P_z^0)}$$

$$\mathcal{M}(z, P_z, P_z^0 = 0) = \sum_n c_n(z^2\mu^2)\langle x^n\rangle(\mu)\frac{(-iP_z z)^n}{n!}$$

$$\Big\downarrow$$

$$\mathcal{M}(z, P_z, P_z^0) = \frac{\sum_n c_n(z^2\mu^2)\langle x^n\rangle(\mu)\frac{(-iP_z z)^n}{n!}}{\sum_n c_n(z^2\mu^2)\langle x^n\rangle(\mu)\frac{(-iP_z^0 z)^n}{n!}};$$

T. Izubuchi, X. Ji, L. Jin, I. W. Stewart, and Y. Zhao, Phys. Rev. D **98**, 056004 (2018), arXiv:1801.03917 [hep-ph].

Matching @ NLO

Resummation in another paper:

X. Gao, K. Lee, S. Mukherjee, and Y. Zhao, Phys. Rev. D **103**, 094504 (2021), arXiv:2102.01101 [hep-ph].

Note: Matching implemented via leading-twist OPE form throughout.
No complication due to non-factorizability for non-zero P^0_z

Leading-twist OPE expression for RI-MOM

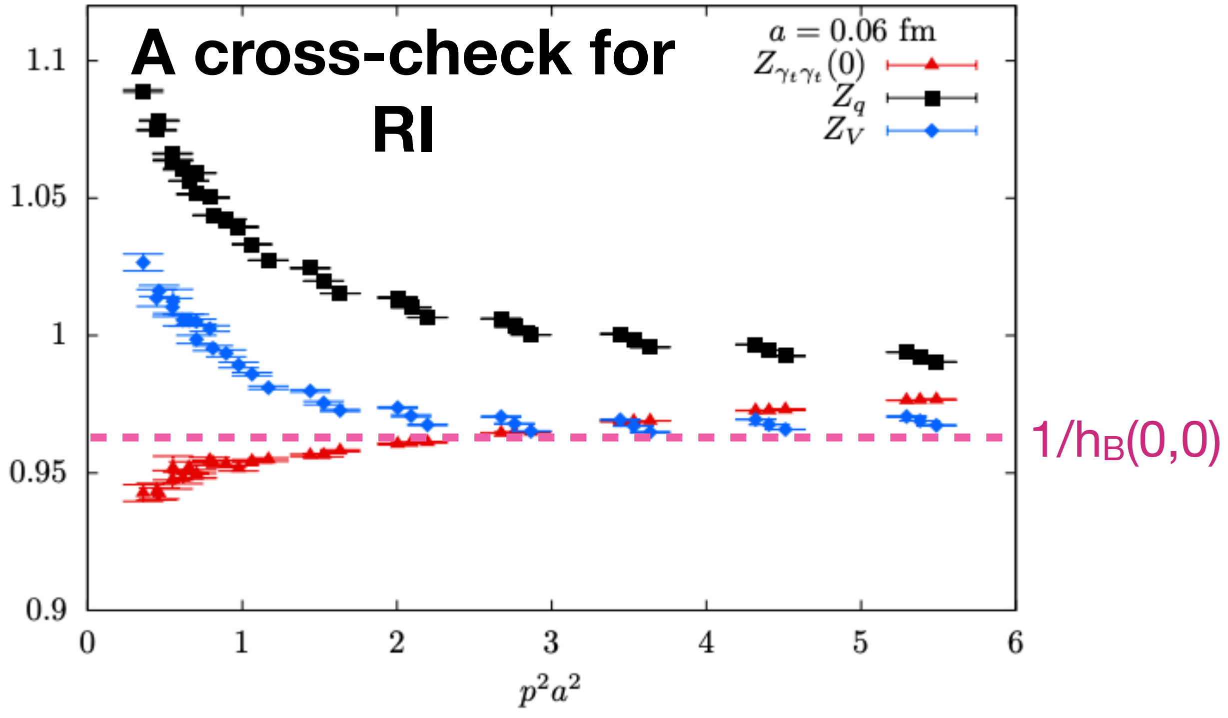
$$h_0^R(z,P_z,P^R) = Z_q Z_{\gamma_t\gamma_t}(z,P^R)\,h^B(z,P_z) \xrightarrow[\text{Ratio}]{\text{Double}} h^R(z,P_z,P^R) \equiv \frac{h_0^R(z,P_z,P^R)}{h_0^R(0,P_z,P^R)}$$

$$\boxed{h^R(z,P_z,P^R) = \sum_n c_n^{\mathrm{RI}}(z^2,\mu^2,P^R)\langle x^n\rangle(\mu)\frac{(-iP_z z)^n}{n!}}$$

$$c_n^{\mathrm{RI}}(z^2,\mu^2,P^R) = \underset{\uparrow}{Z_{\mathrm{ratio}\to\mathrm{RI}}(z,P^R,\mu)}\,c_n\underset{\uparrow}{(}z^2\mu^2),$$

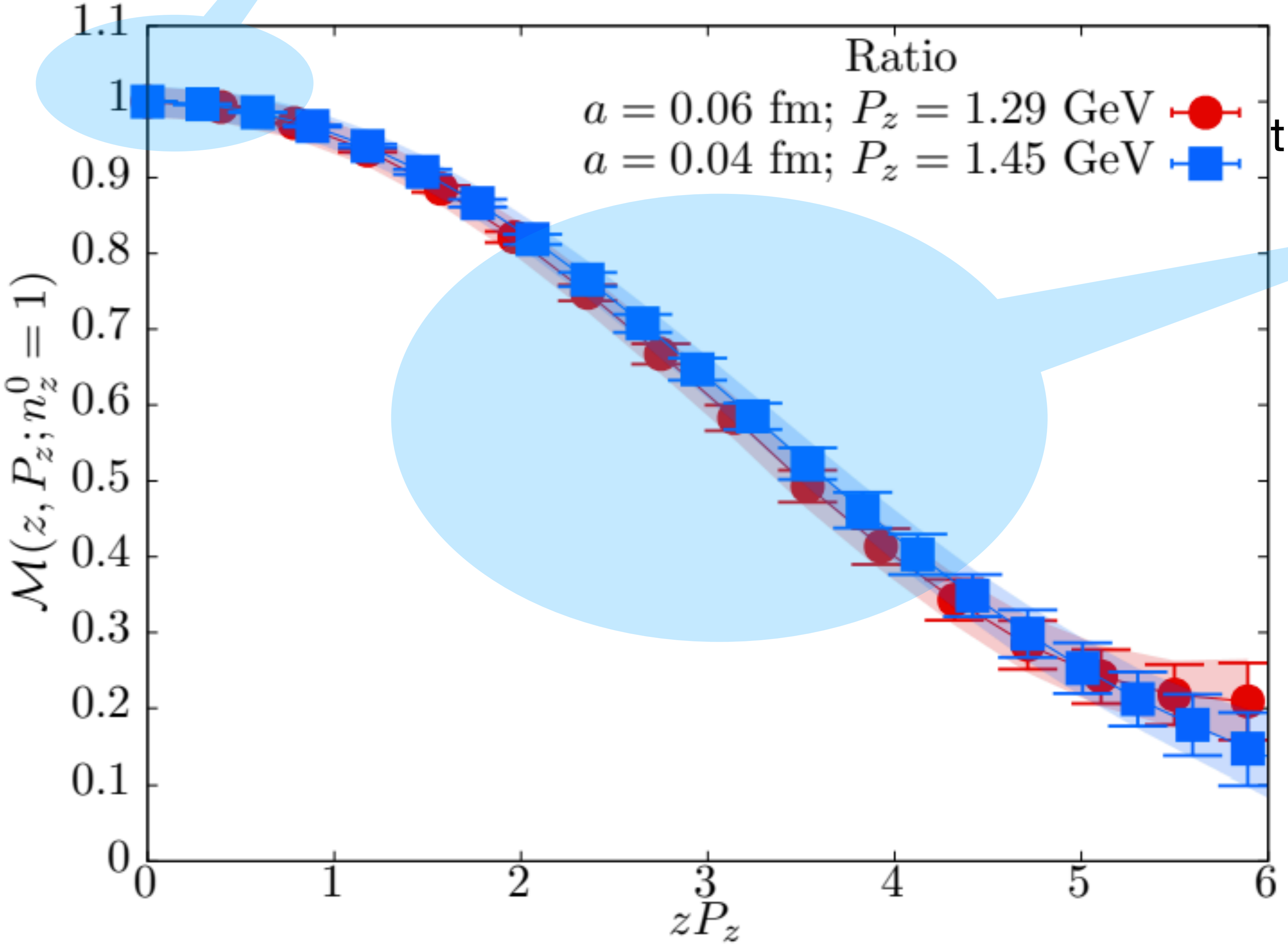
1-loop
Conversion factor

Ratio C_n



Analysis of lattice artifacts

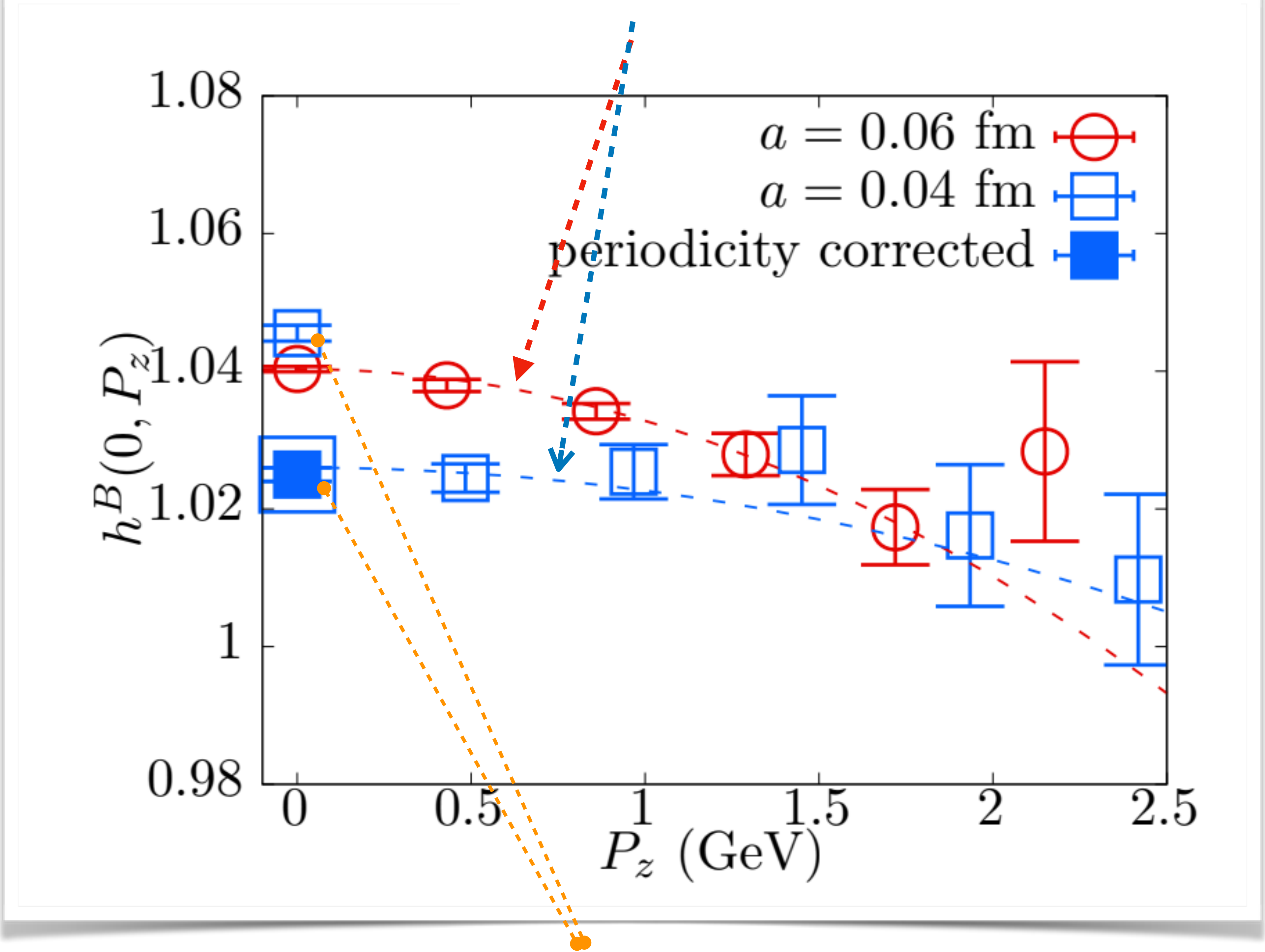
Lattice corrections hidden here;
Important because errors are too tiny after double ratio



Within errors,
the data are consistent
In the intermediate z

Let us look at the bare z=0 local current operator matrix element before double ratios

$$h^B(z=0,P_z) = h^B(z=0,P_z=0) + b(P_z a)^2$$

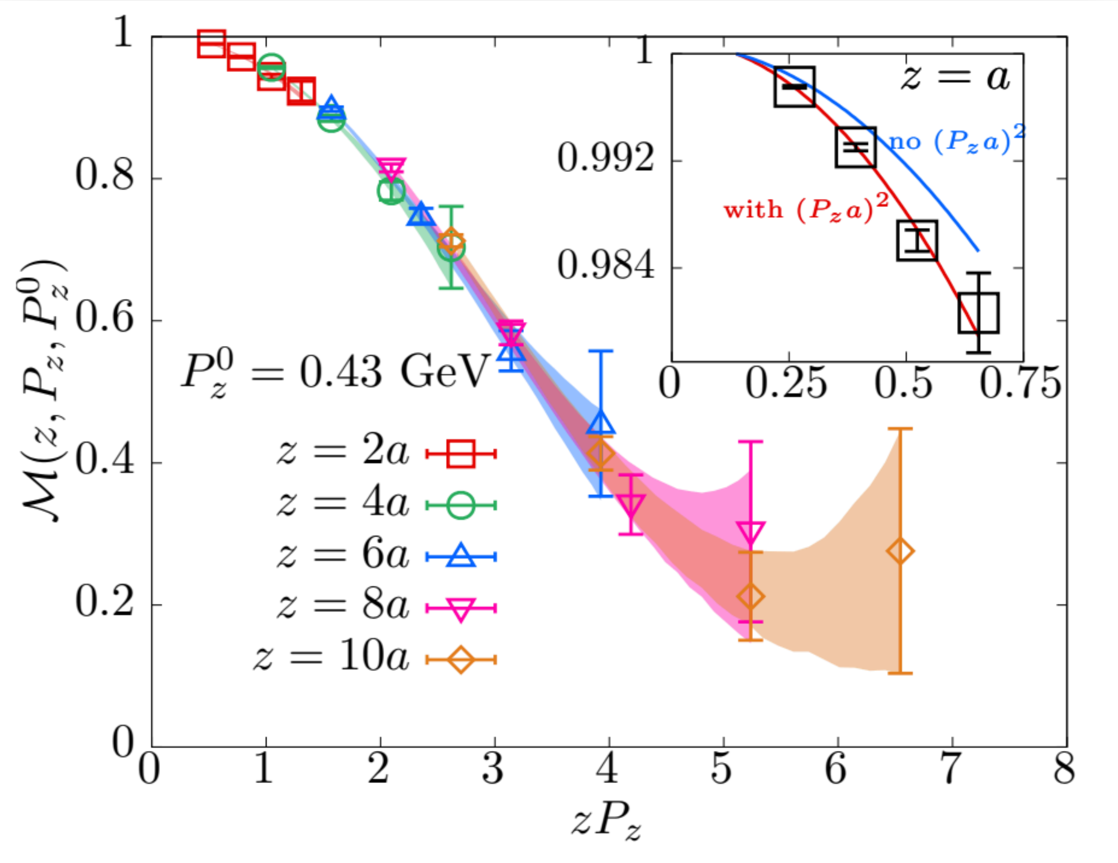


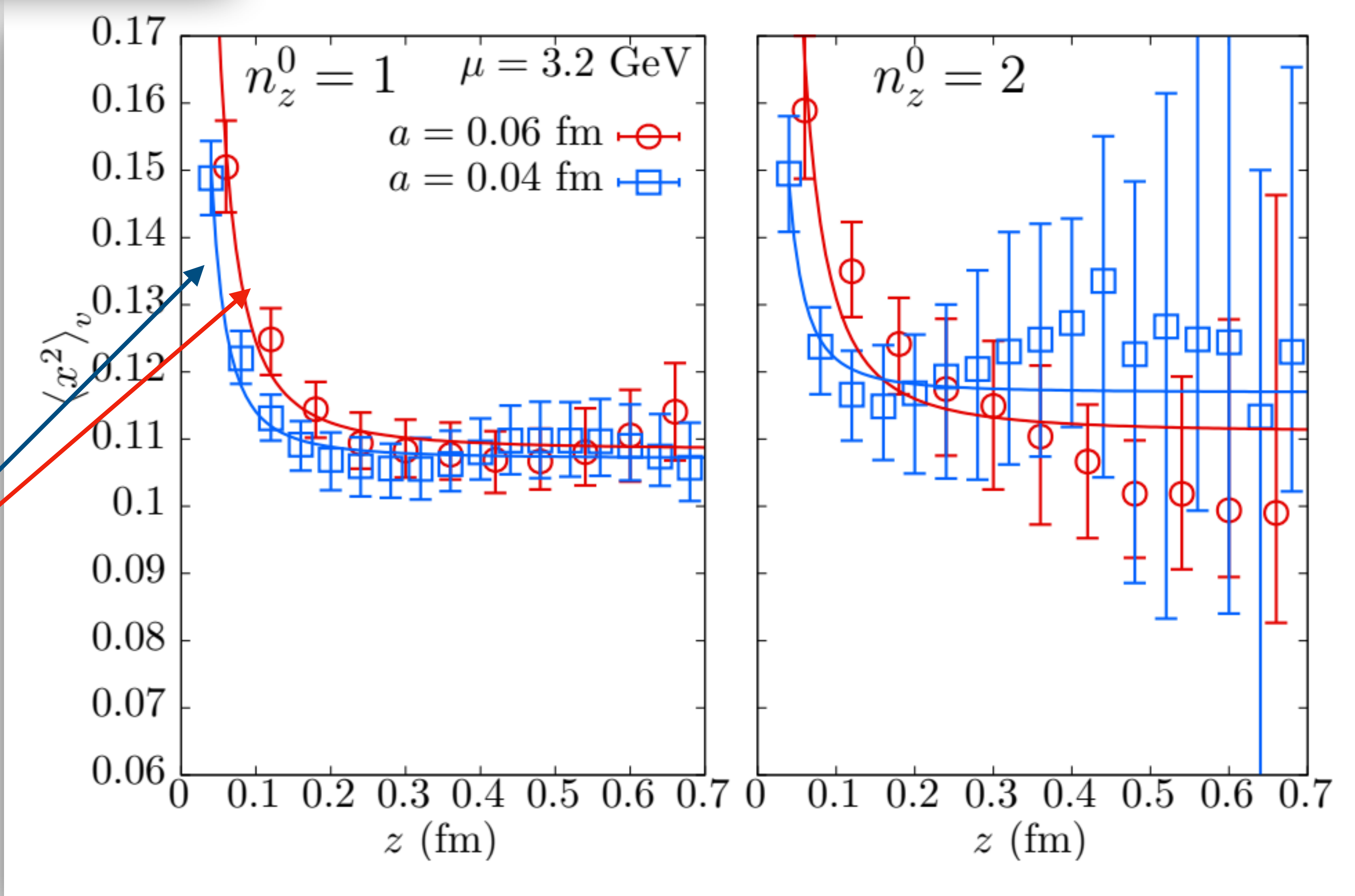
Wrap-around effect in 3pt

Analysis of lattice artifacts via “OPE without OPE”

Fit <x^n> via leading-twist OPE as a **function of P$_z$ at fixed z**

J. Karpie, K. Orginos, and S. Zafeiropoulos, JHEP **11**, 178 (2018), arXiv:1807.10933 [hep-lat].





$$\langle x^2 \rangle + 2r(z/a)^{-2}$$

Inference:
presence of momentum dependent lattice corrections

$$\mathcal{M}(z, P_z, P_z^0) = \frac{\sum_n c_n(z^2\mu^2)\langle x^n\rangle(\mu)\frac{(-iP_z z)^n}{n!} + r(aP_z)^2}{\sum_n c_n(z^2\mu^2)\langle x^n\rangle(\mu)\frac{(-iP_z^0 z)^n}{n!} + r(aP_z^0)^2}$$

↓

$$\langle x^2\rangle \to \langle x^2\rangle - \frac{2r}{c_2(\mu^2 z^2)}\frac{1}{(z/a)^2}$$

cross-check: r= 0.021 and 0.022 on the 0.04 fm and 0.06 fm

A similar observation with (a/z) correction in nucleon in a recent paper

J. Karpie, K. Orginos, A. Radyushkin, and S. Zafeiropoulos, (2021), arXiv:2105.13313 [hep-lat].

Analysis of higher-twist corrections

Internal consistency of 1-loop twist-2 framework

RI-MOM

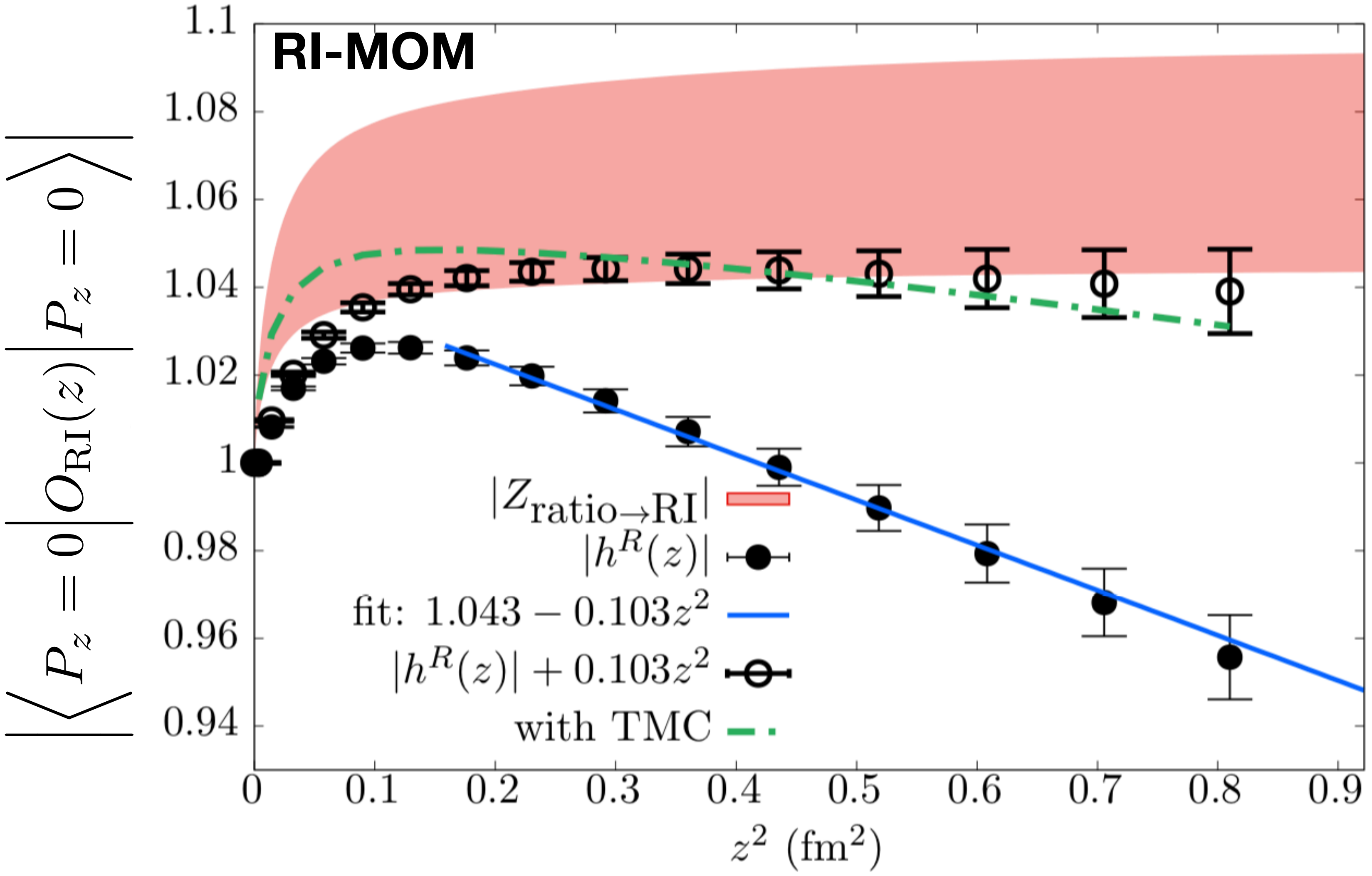
$$O_{\mathrm{RI}}(z) = Z(p^R, z) O_{\mathrm{bare}}(z)$$

$$\Big\downarrow \; Z_{\mathrm{conversion}}(z)$$

Ratio

$$\frac{\left\langle P_z \middle| O_{\mathrm{bare}}(z) \middle| P_z \right\rangle}{\left\langle P_z^0 \middle| O_{\mathrm{bare}}(z) \middle| P_z^0 \right\rangle}$$

Internal consistency of 1-loop twist-2 framework



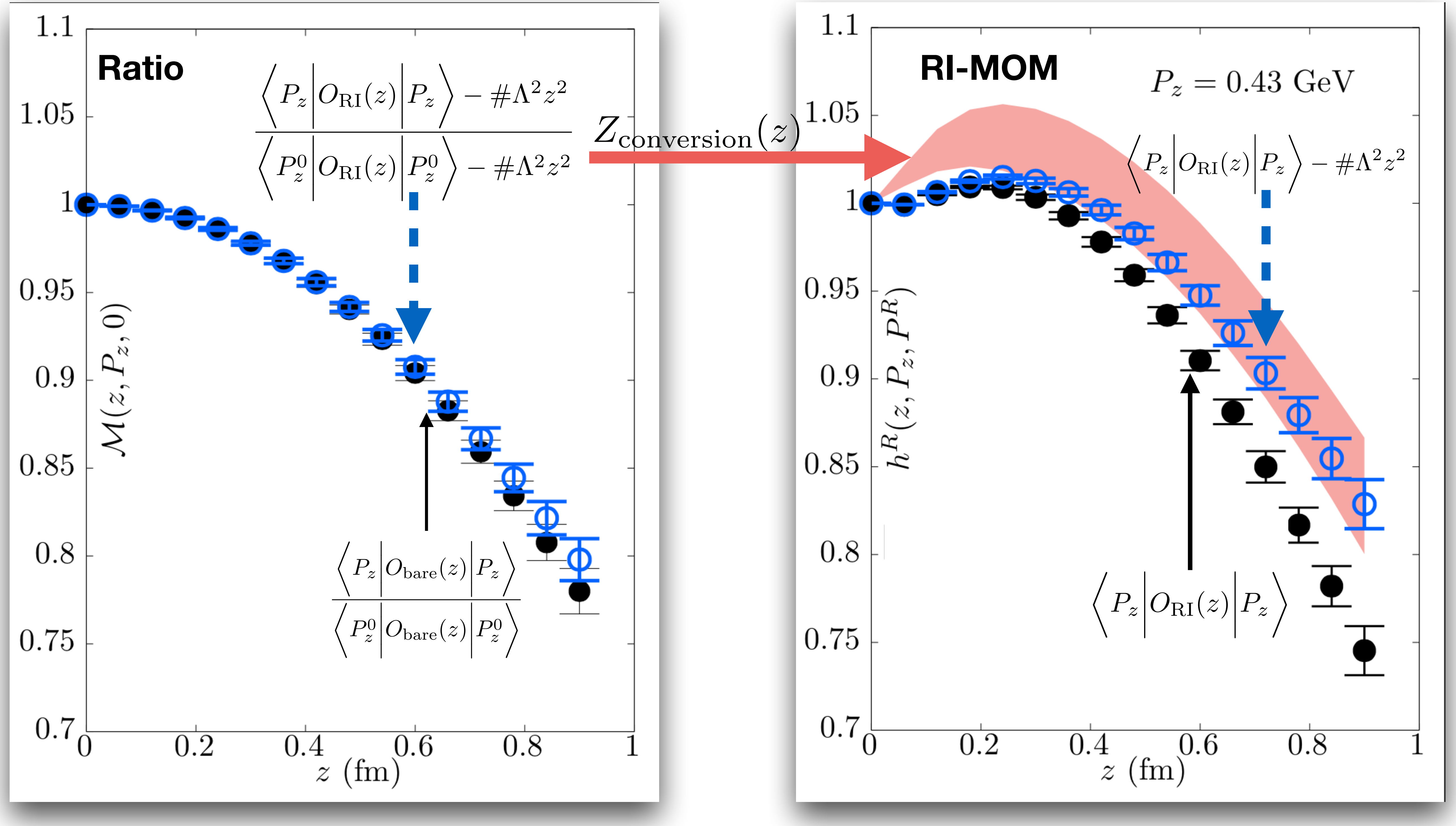
Zero momentum matrix element
is a great place to study
higher twist effects!

$$\mathcal{M}(z, P_z = 0) = c_0(z^2\mu^2) + \sum_{n=1} \xcancel{\#(izP_z)^n} \langle x^n \rangle$$

$$+ \#z^2\Lambda^2_{\mathrm{QCD}} + \ldots$$

Internal consistency of 1-loop twist-2 framework

Smallest non-zero P_z



Analysis of moments:

Fit leading-twist OPE with moments as fit parameters

Fit data satisfying $z \in \left[z_{\min}, z_{\max}\right]$ **and** $P_z > P_z^0$ **with <x^n> as fit parameters**

$$\mathcal{M}(z, P_z, P_z^0) = \frac{\sum_n c_n(z^2\mu^2)\langle x^n\rangle(\mu)\frac{(-iP_z z)^n}{n!} + r(aP_z)^2}{\sum_n c_n(z^2\mu^2)\langle x^n\rangle(\mu)\frac{(-iP_z^0 z)^n}{n!} + r(aP_z^0)^2}$$

Technical aside:

Positivity of u-d valence PDF

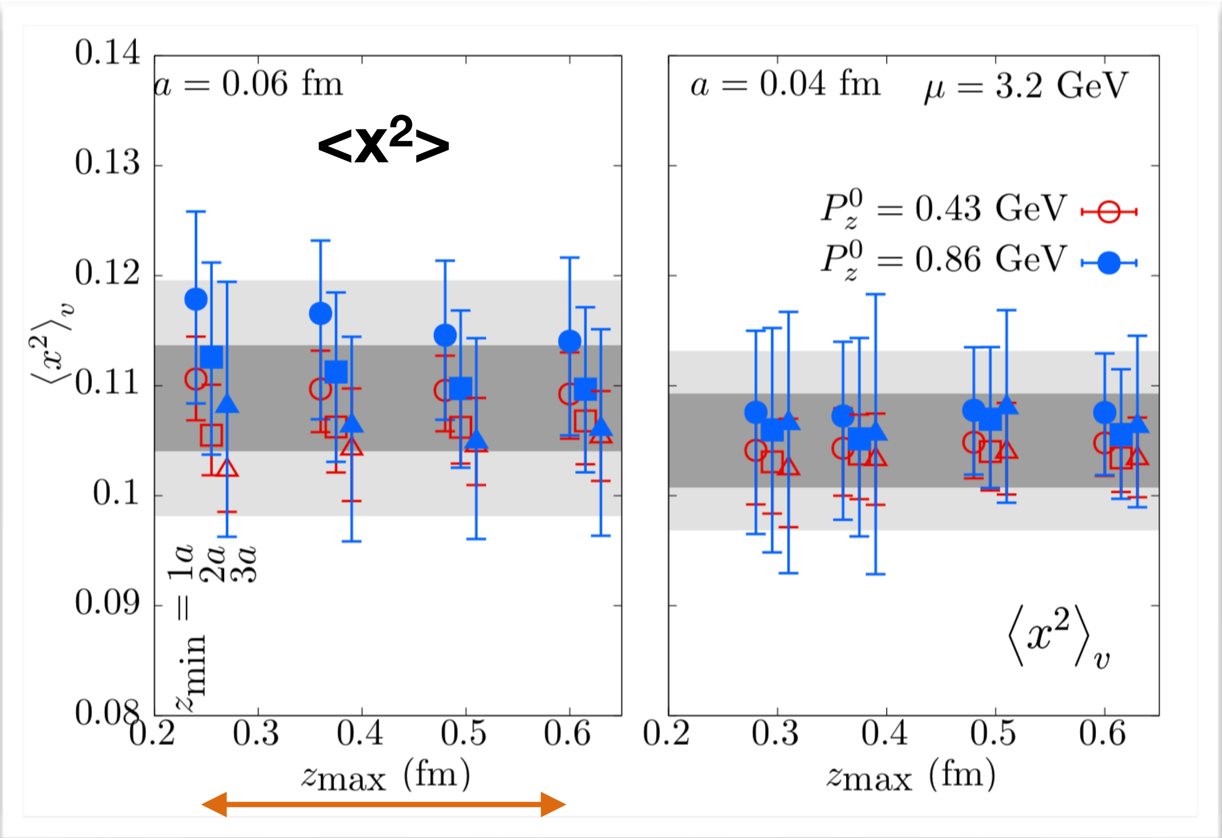
↓ Condition on derivative of <x^n> wrt n

$$\langle x^{n+2}\rangle_{u-d} < \langle x^n\rangle_{u-d} \quad \text{and,}$$
$$\langle x^{n+2}\rangle_{u-d} + \langle x^{n-2}\rangle_{u-d} - 2\langle x^n\rangle_{u-d} > 0.$$

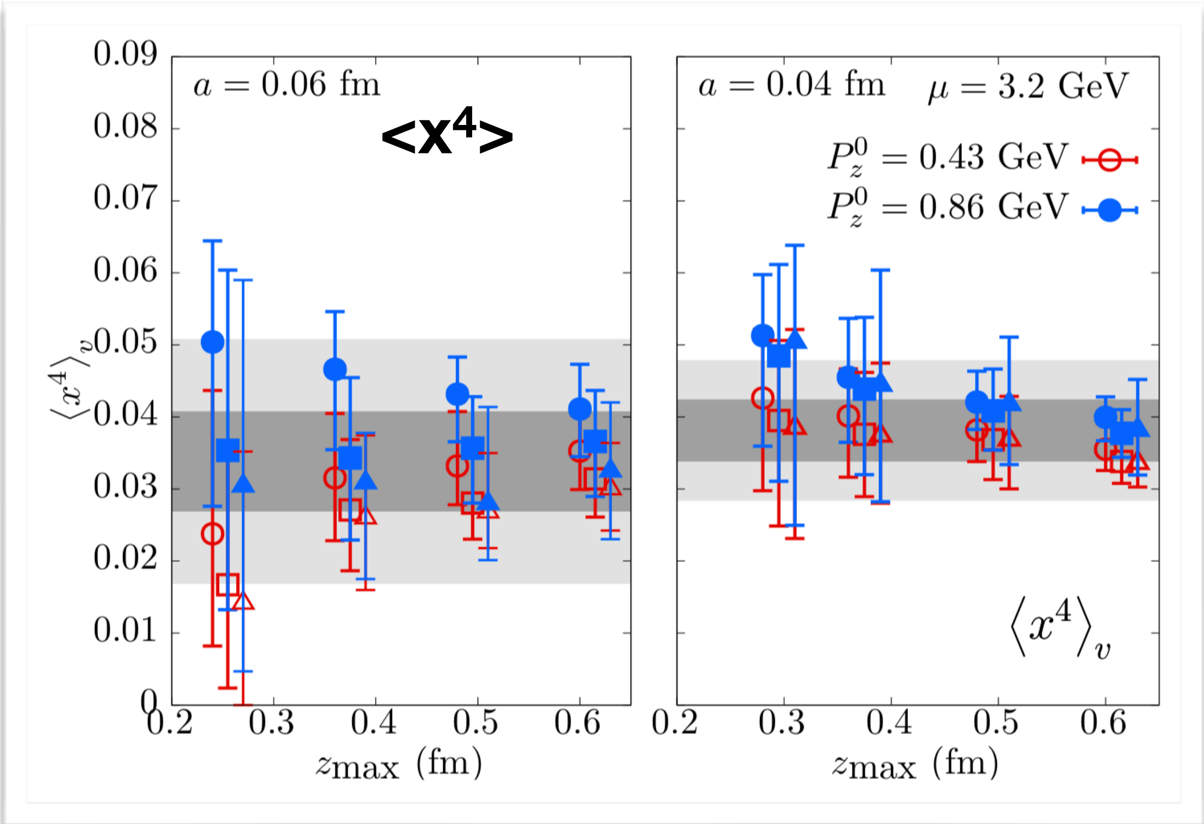
↓ A convenient variable change

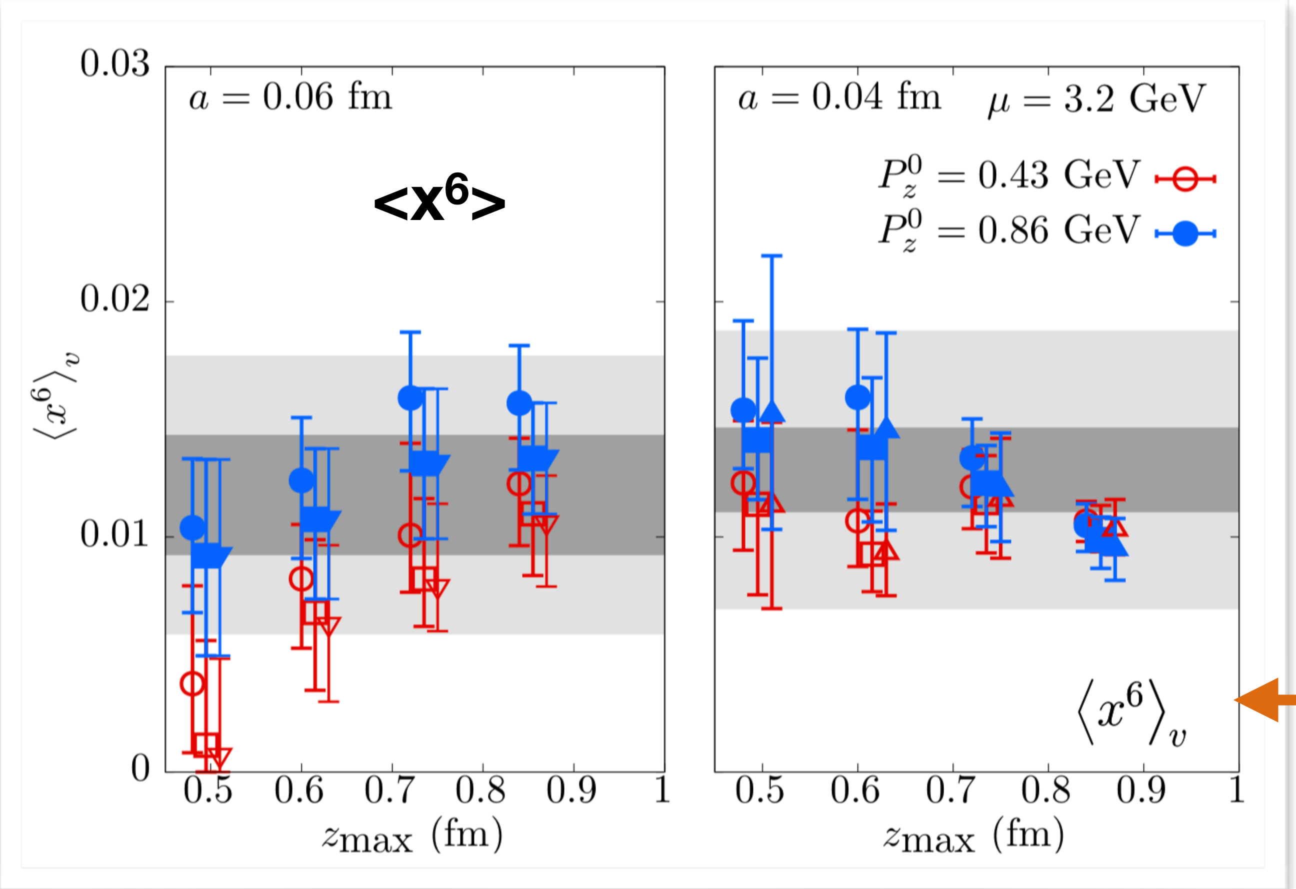
$$\langle x^n\rangle_v \equiv \sum_{i=n}^{N}\sum_{j=i}^{N} e^{-\lambda_j},$$

Direct computation of even valence moments



Maximal z used in fits to twist-2 OPE





Need larger z for higher moments.
Prior on lower moment helps.

$$\chi^2 = \chi^2 + \sum_{i=1}^{N_{\text{prior}}} \frac{(\langle x^i\rangle_v - \langle x^i\rangle_{\text{prior}})^2}{(\sigma_i^{\text{prior}})^2}$$

Determination of x-dependent PDFs

Fit PDF-Ansatze to lattice matrix elements

Fitting, approximations and their rationale

Minimize chi-square: fit a finite range of z and z P_z

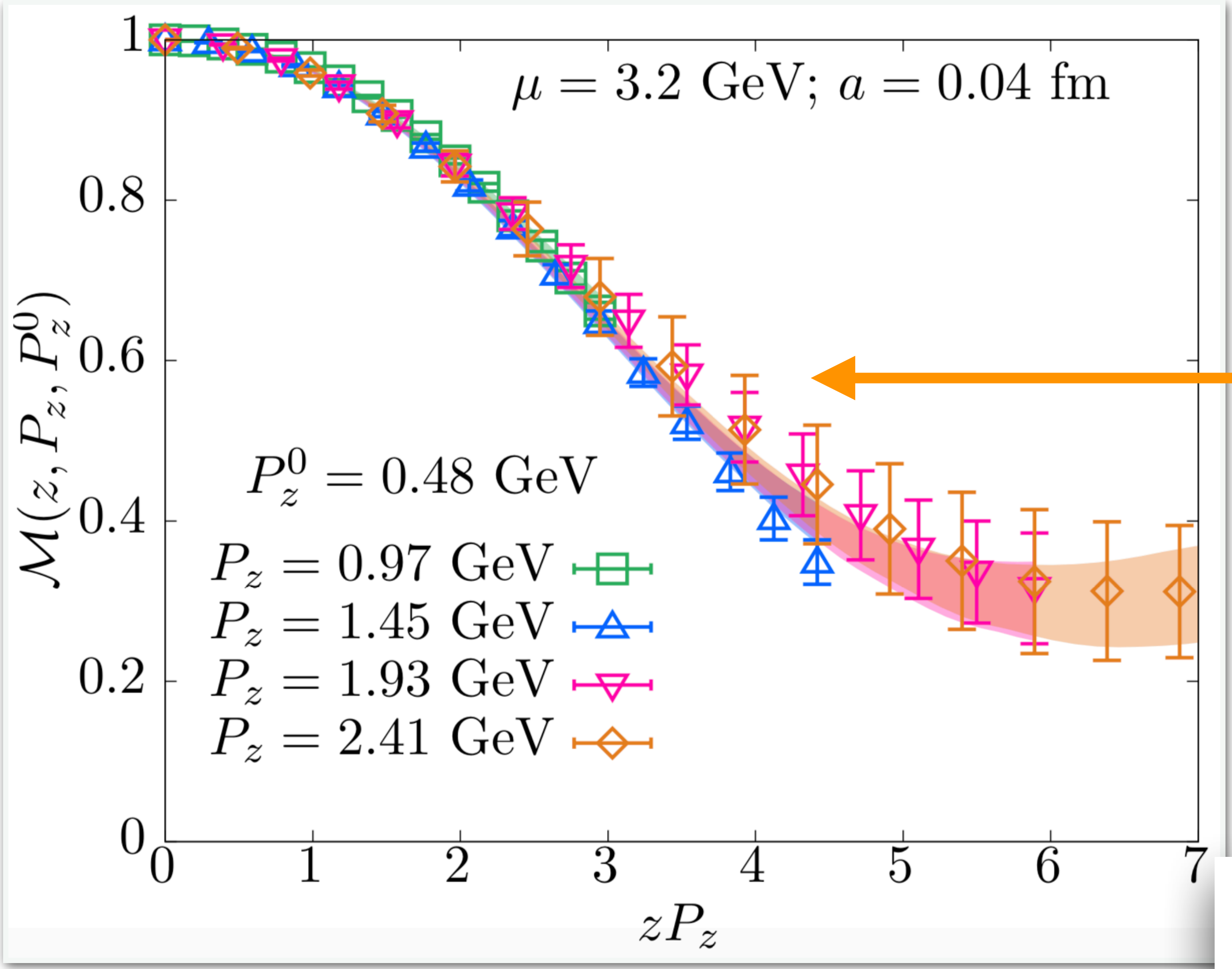
$$\chi^2 \equiv \sum_{P_z > P_z^0}^{P_z^{\max}} \sum_{z=z_1}^{z_2} \frac{\left(\mathcal{M}(z, P_z, P_z^0) - \mathcal{M}_{\text{model}}(z, P_z, P_z^0; \alpha, \ldots)\right)^2}{\sigma^2_{\text{stat}}(z, P_z, P_z^0) + \sigma^2_{\text{sys}}(z, P_z, P_z^0)}$$

Include uncertainty in perturbation theory added in quadrature:

$$\sigma_{\text{sys}}(z, \ldots) = \frac{1}{2}\left(\mathcal{M}_{\text{model}}(z, \ldots)|_{\alpha_s(\mu/2)} - \mathcal{M}_{\text{model}}(z, \ldots)|_{\alpha_s(2\mu)}\right).$$

Drawback: Correlation very important! One needs to weigh-in two things:

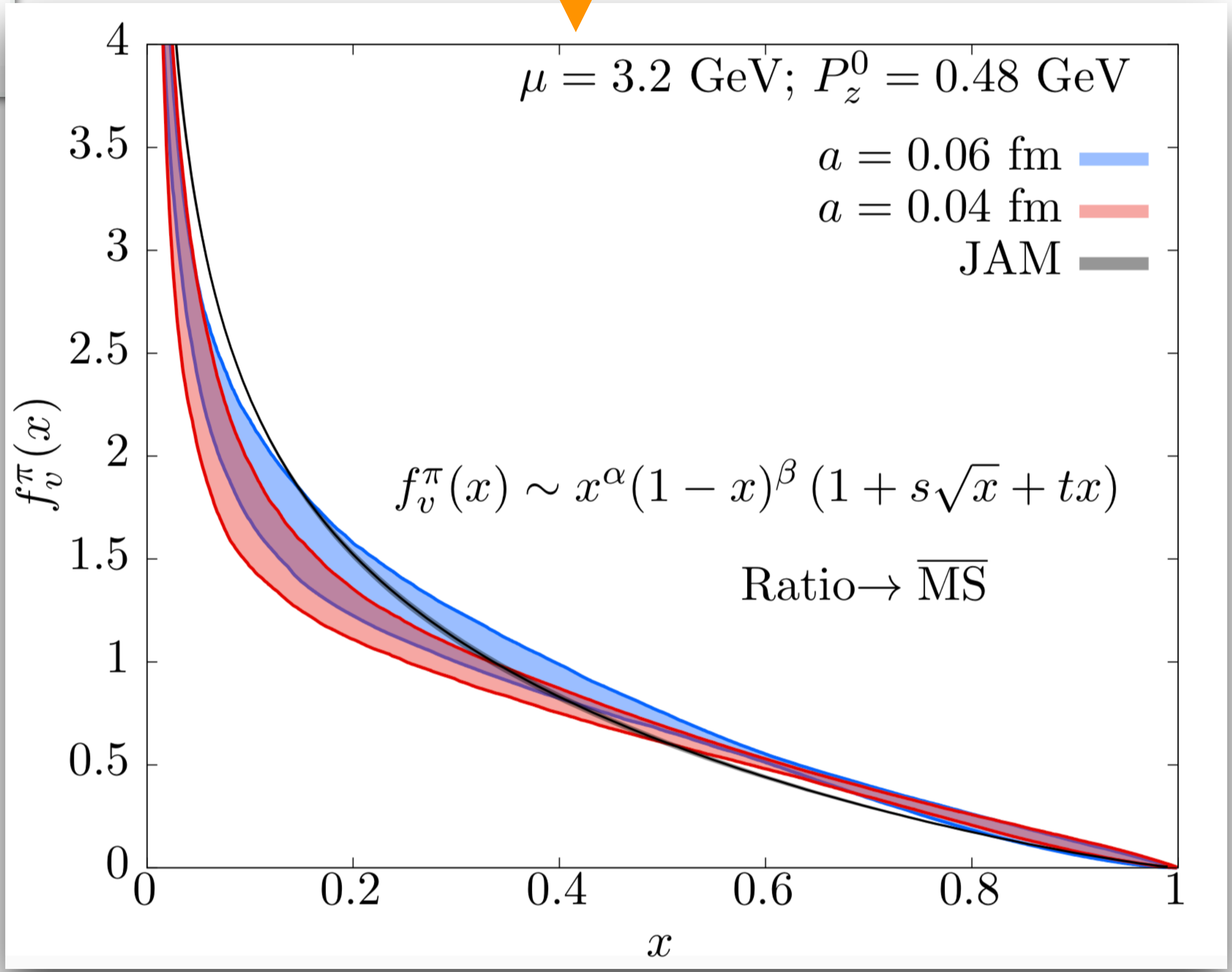
- Scale dependence in matching is comparable to statistical errors. Can’t ignore.
- Model selection when underlying theory has larger uncertainty: how to do it?



Fit ITD to model PDFs

$$f_v^\pi(x,\mu) = \mathcal{N} x^\alpha (1-x)^\beta \left(1 + s\sqrt{x} + tx\right)$$

Corresponding PDF

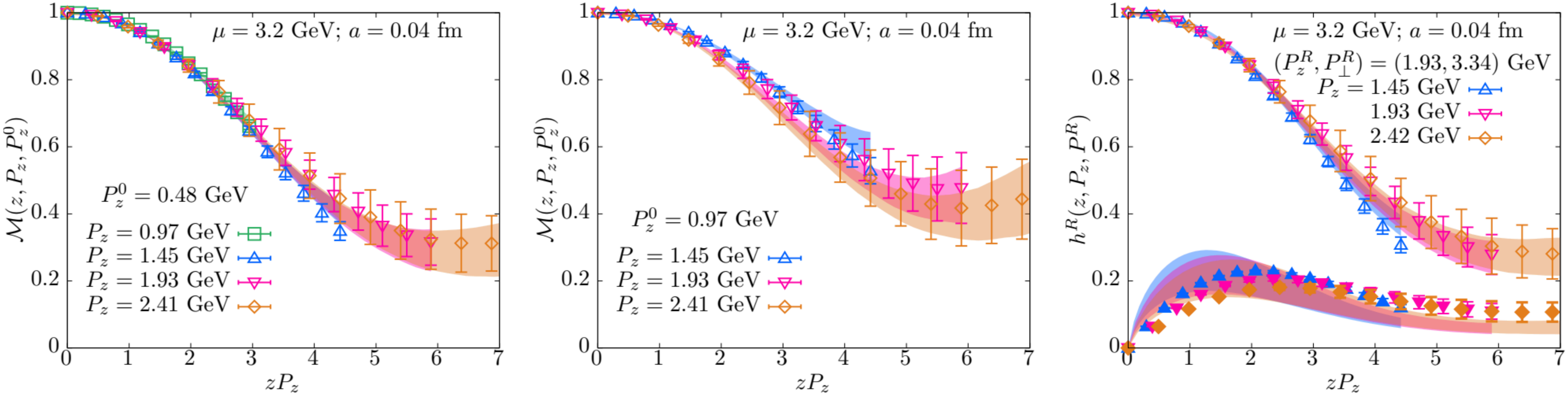


Reconstructing PDFs

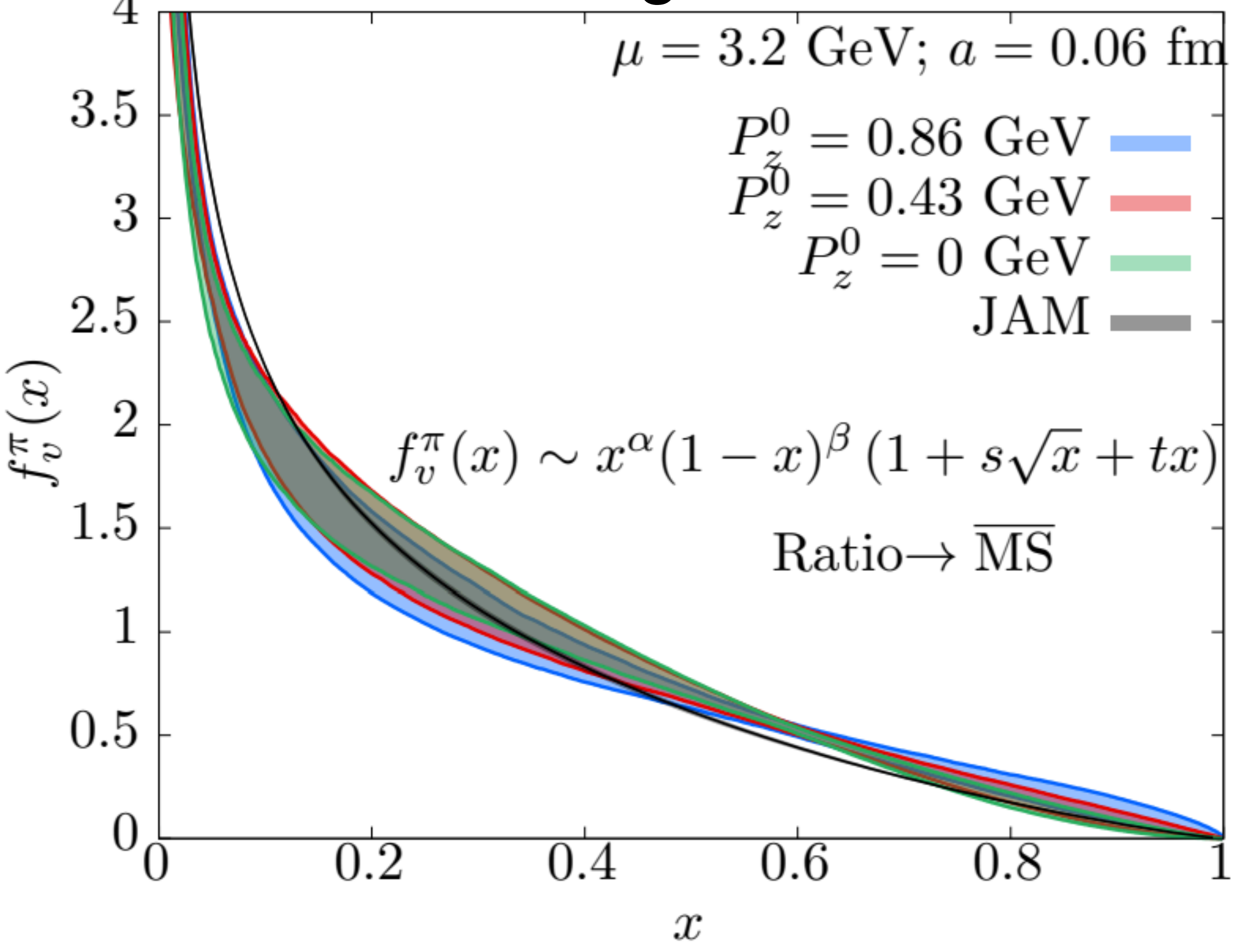
Fit only data points with $z \in [1(\text{ or } 2)a, z_{\max}]$

0.32 fm < $Z_{\max}$ < 0.72 fm

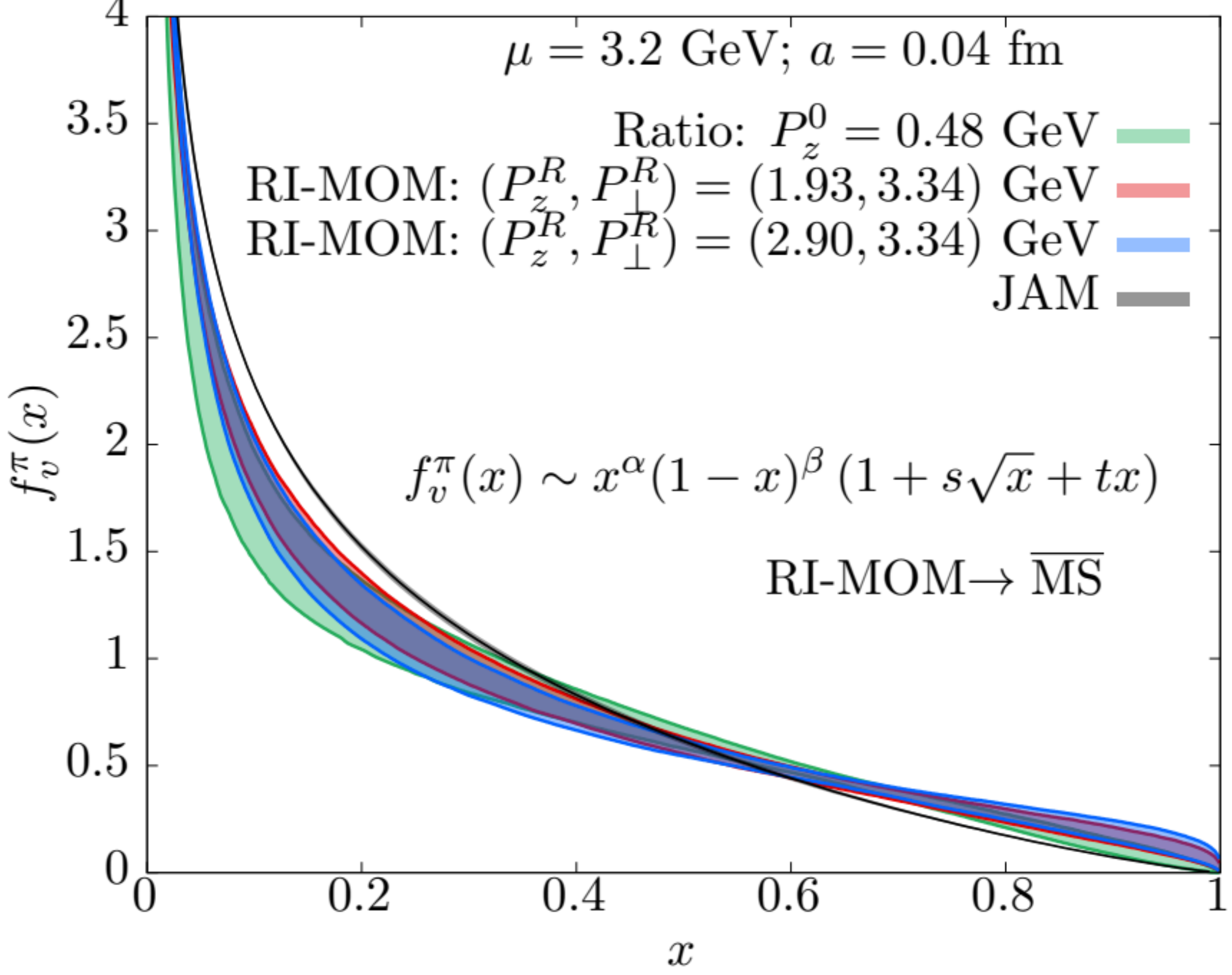
Renormalization scheme dependence?



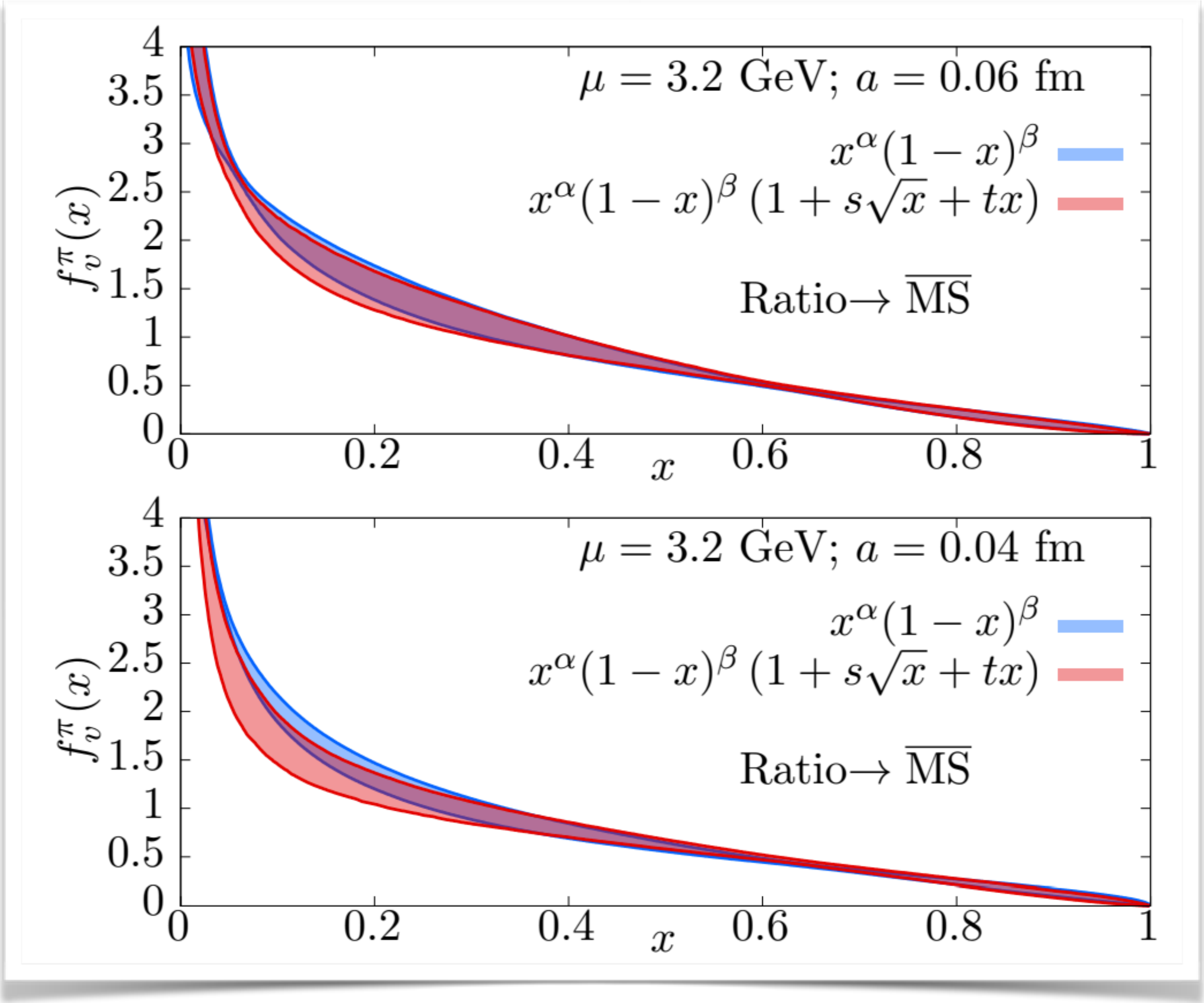
Variation among ratio definitions



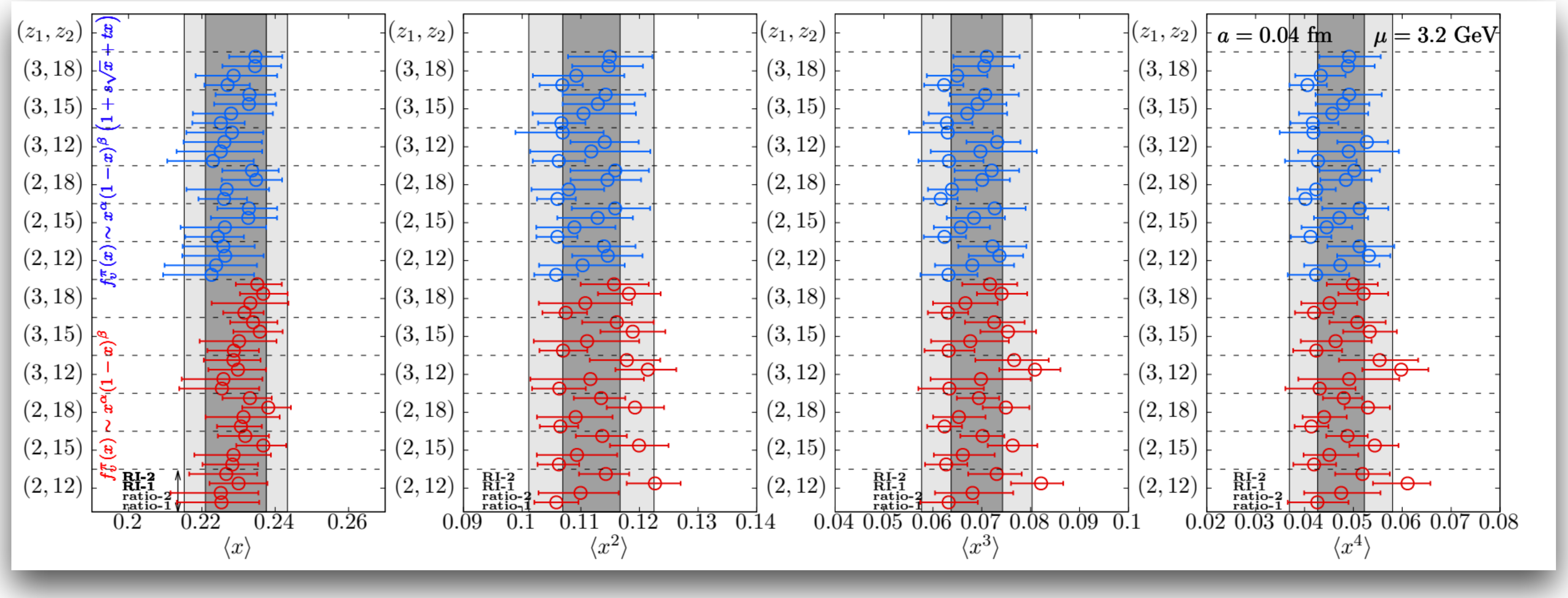
Variation from ratio to RI



Ansatz dependence?



Quantifying the fit systematics



In each bootstrap sample of some quantity “A”:

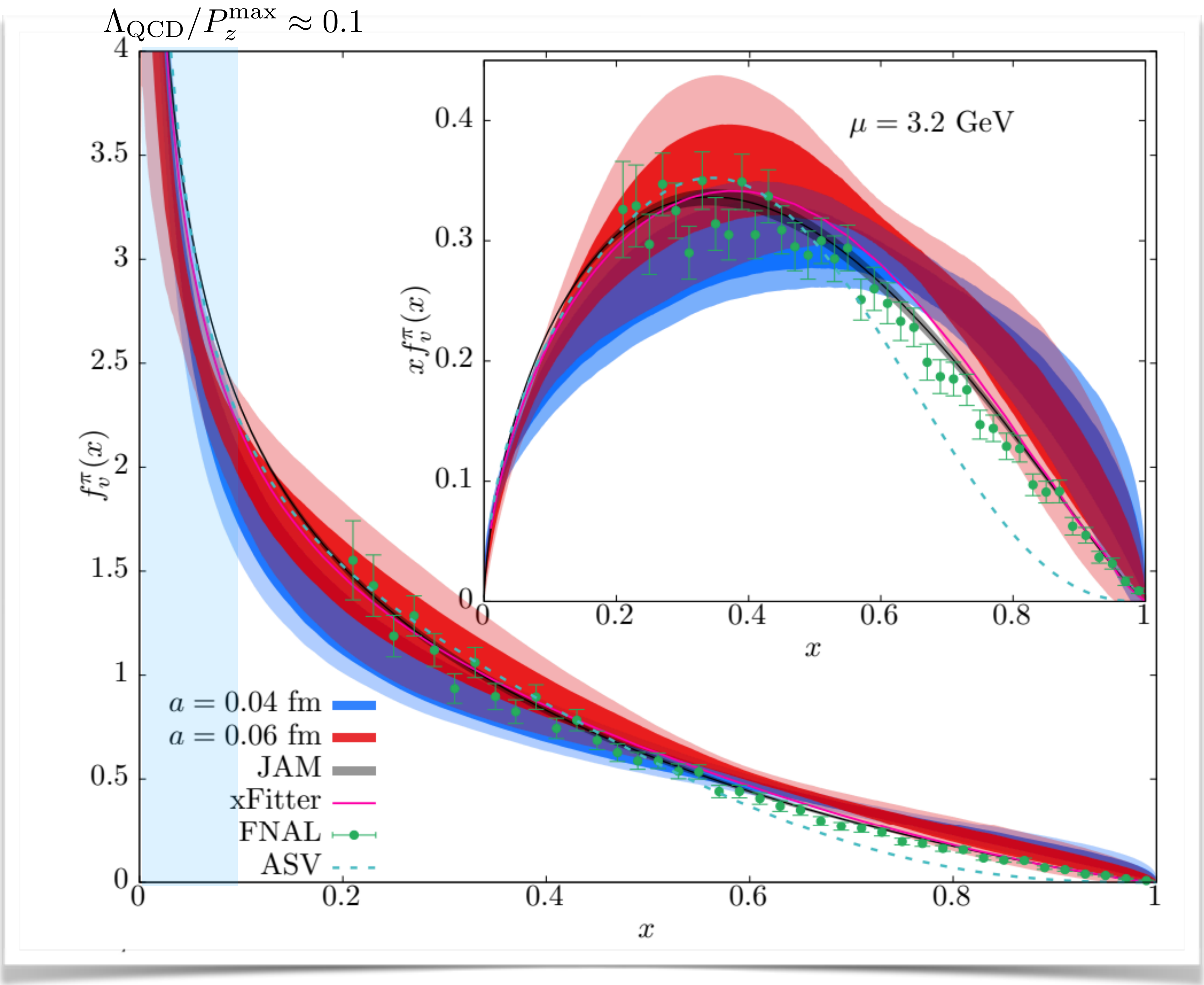
A = Mean[{ set of estimates over choice of z-range, renormalization scheme, ansatz dependence**}]**

sys-error(A) =

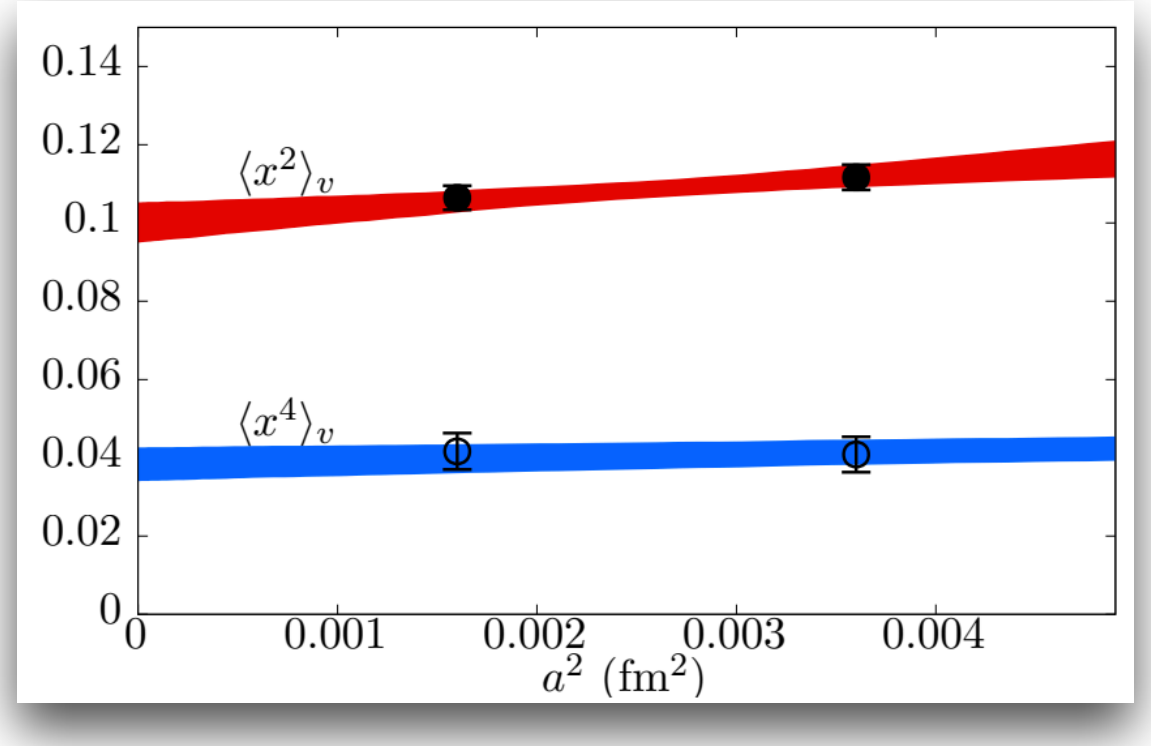
SD[{ set of estimates over choice of z-range, renormalization scheme, ansatz dependence**}]**

Bootstrap estimate of A : mean and statistical error

PDF estimate with statistical and systematic error included

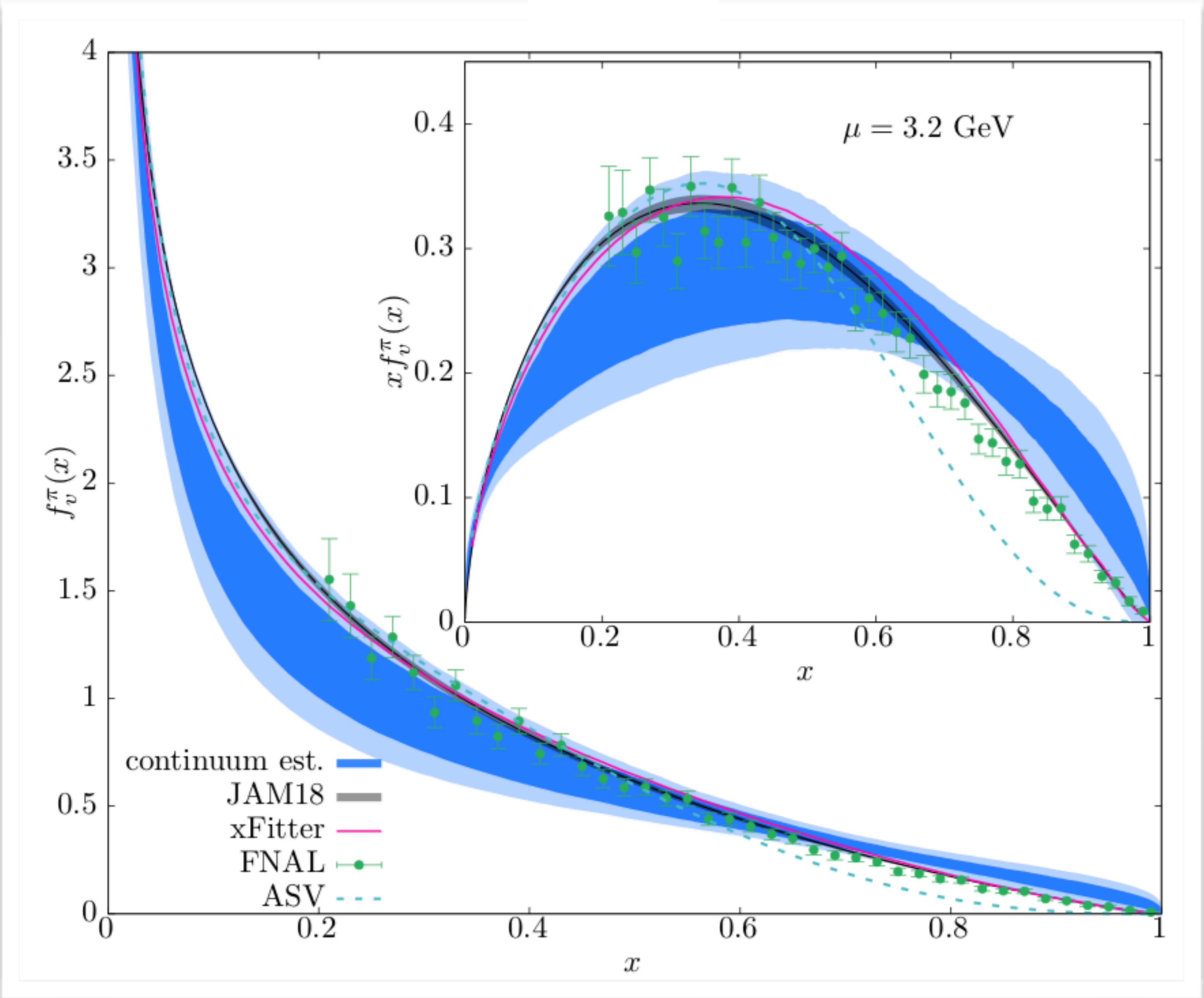


What to expect in continuum limit?



Try fits with twist-2 OPE using a-dependent moments:

$$\langle x^n\rangle_v(a) = \langle x^n\rangle_v + d_n a^2$$

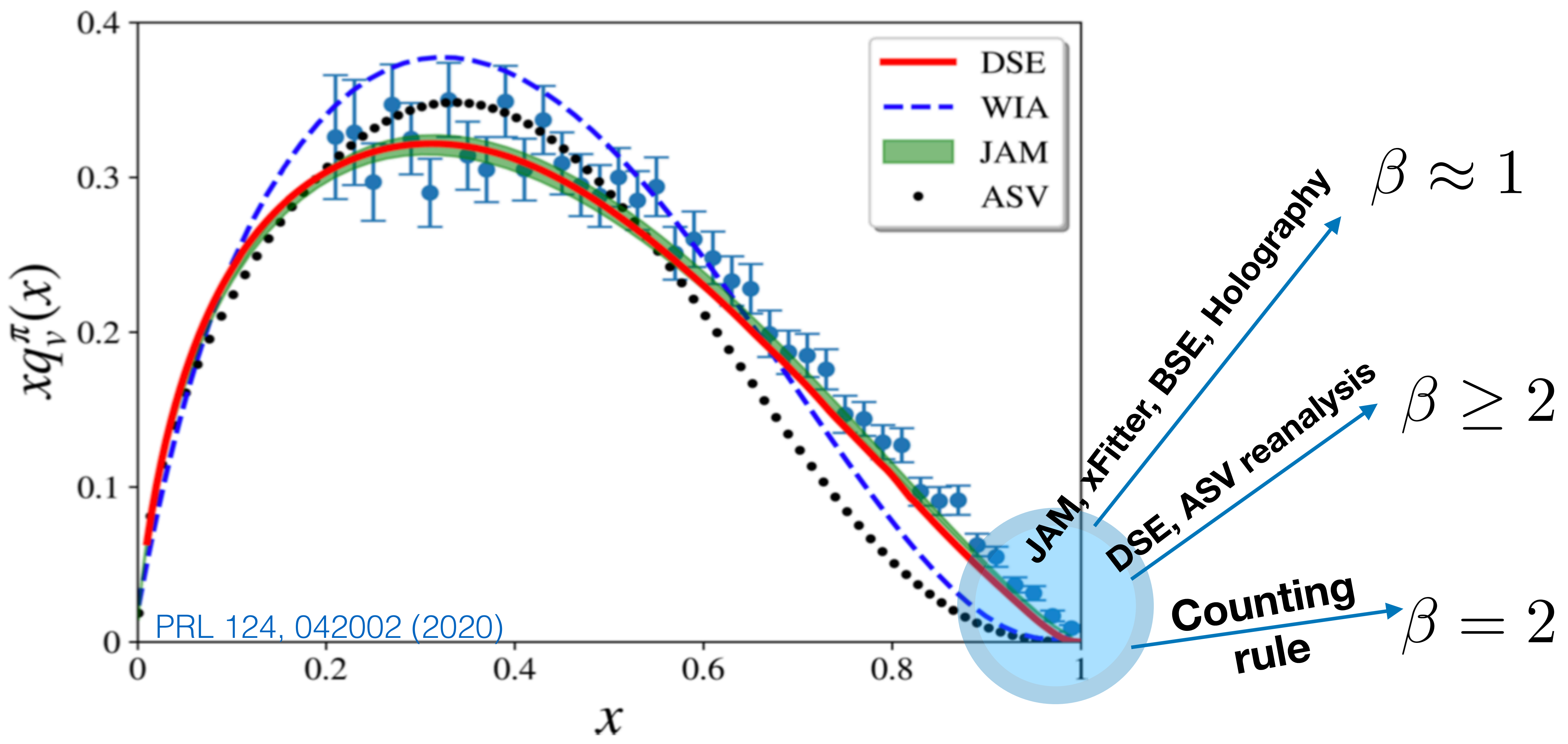


- Two fine lattice spacing with a² corrections
- Systematic errors: range of quark-antiquark distance, ren. scheme, PDF ansatz..

Large-x behavior of pion: an unresolved issue

Key physics issue is x=1 behavior:

$$\lim_{x\to 1} f_v(x) \sim (1-x)^{\beta}$$



PRL 124, 042002 (2020)

Models of pion PDF respecting NG boson properties predict β > 2

Review in, C. D. Roberts et al, 2102.01765

A wide scatter in β ➡ First principle calculation essential

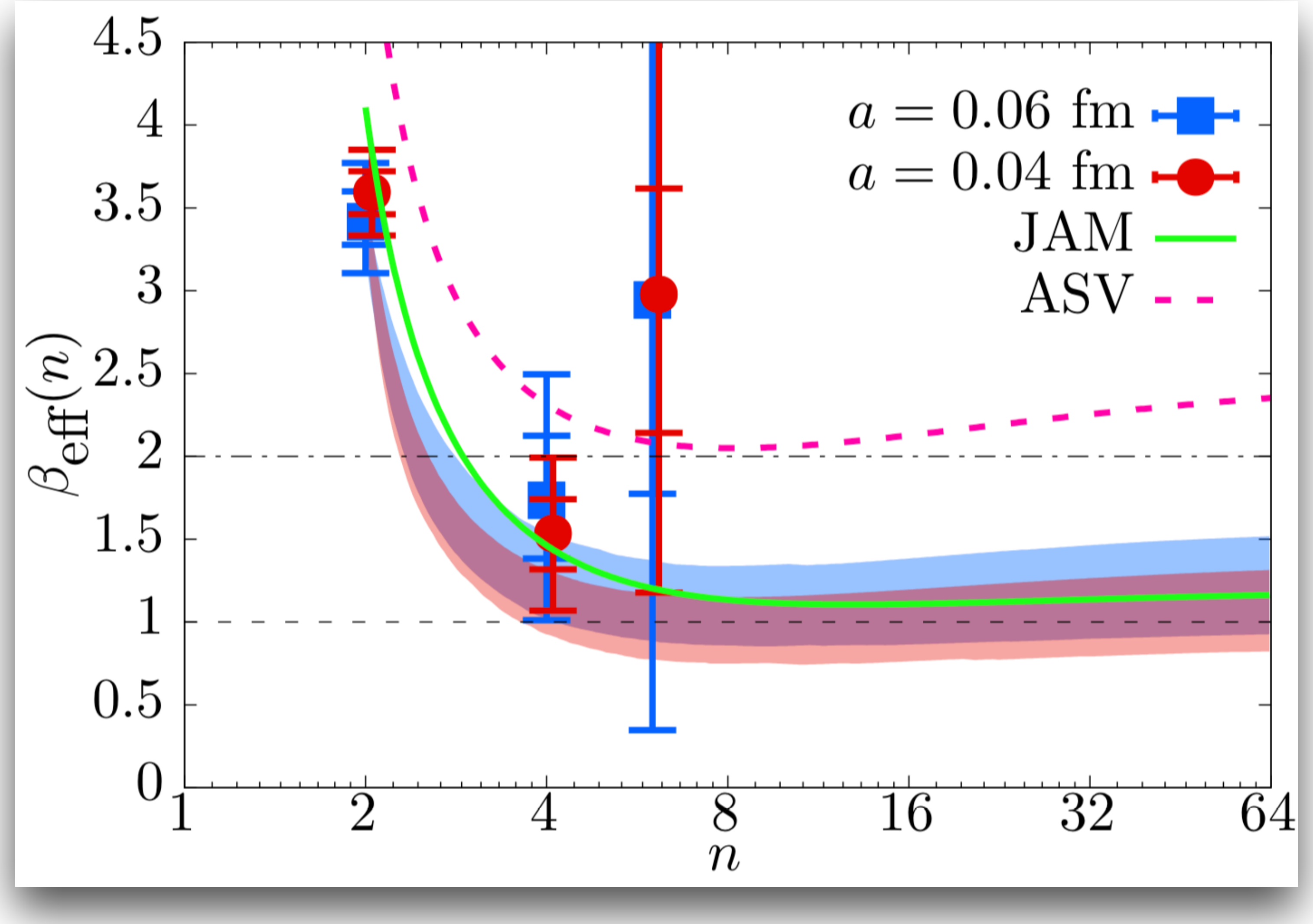
Large-x $(1-x)^\beta$ behavior and lattice QCD

Model independent approach:

$$\langle x^n \rangle \propto n^{-\beta-1}\,(1+\mathcal{O}(1/n))\,.$$

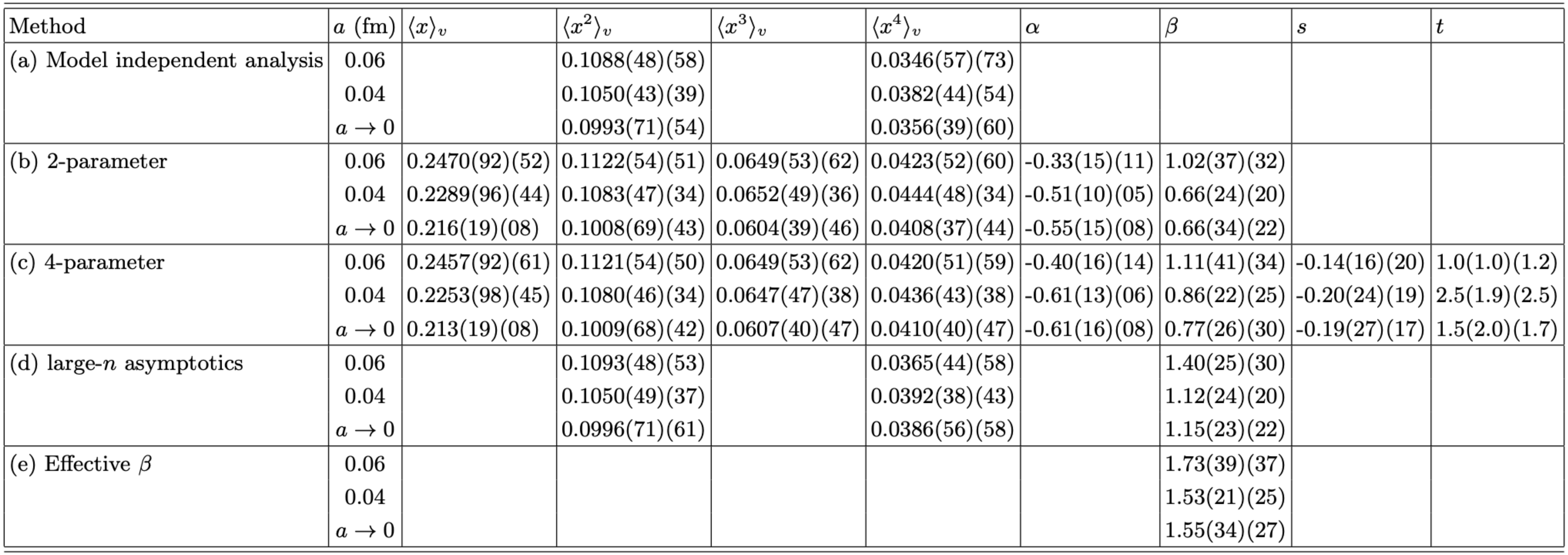
★ Note the universal exponent of large moments

$$\beta_{\text{eff}}(n) \equiv -1 + \frac{\langle x^{n-2}\rangle - \langle x^{n+2}\rangle}{\langle x^n\rangle}\,\frac{n}{4}\,.$$



Up-shot: a semi-anstaz dependent OPE fits using $\langle x^n\rangle_v \equiv n^{-\beta}\left(\frac{A_0}{n} + \frac{A_1}{n^2} + \frac{A_2}{n^3}\right)$

To summarize…



Method	a (fm)	$\langle x\rangle_v$	$\langle x^2\rangle_v$	$\langle x^3\rangle_v$	$\langle x^4\rangle_v$	α	β	s	t
(a) Model independent analysis	0.06		0.1088(48)(58)		0.0346(57)(73)				
	0.04		0.1050(43)(39)		0.0382(44)(54)				
	$a \to 0$		0.0993(71)(54)		0.0356(39)(60)				
(b) 2-parameter	0.06	0.2470(92)(52)	0.1122(54)(51)	0.0649(53)(62)	0.0423(52)(60)	-0.33(15)(11)	1.02(37)(32)		
	0.04	0.2289(96)(44)	0.1083(47)(34)	0.0652(49)(36)	0.0444(48)(34)	-0.51(10)(05)	0.66(24)(20)		
	$a \to 0$	0.216(19)(08)	0.1008(69)(43)	0.0604(39)(46)	0.0408(37)(44)	-0.55(15)(08)	0.66(34)(22)		
(c) 4-parameter	0.06	0.2457(92)(61)	0.1121(54)(50)	0.0649(53)(62)	0.0420(51)(59)	-0.40(16)(14)	1.11(41)(34)	-0.14(16)(20)	1.0(1.0)(1.2)
	0.04	0.2253(98)(45)	0.1080(46)(34)	0.0647(47)(38)	0.0436(43)(38)	-0.61(13)(06)	0.86(22)(25)	-0.20(24)(19)	2.5(1.9)(2.5)
	$a \to 0$	0.213(19)(08)	0.1009(68)(42)	0.0607(40)(47)	0.0410(40)(47)	-0.61(16)(08)	0.77(26)(30)	-0.19(27)(17)	1.5(2.0)(1.7)
(d) large-n asymptotics	0.06		0.1093(48)(53)		0.0365(44)(58)		1.40(25)(30)		
	0.04		0.1050(49)(37)		0.0392(38)(43)		1.12(24)(20)		
	$a \to 0$		0.0996(71)(61)		0.0386(56)(58)		1.15(23)(22)		
(e) Effective β	0.06						1.73(39)(37)		
	0.04						1.53(21)(25)		
	$a \to 0$						1.55(34)(27)		